Solution to Problem 4.12

Note that, as derived in Section 4.5.1, the mean queueing delay $W_{q(k)}$ of customers of priority class k will be given by

$$W_{q(k)} = \frac{R}{(1-r_n-....-r_{k+1})(1-r_n-....-r_k)}$$

This may be rewritten as

$$W_{q(k)} = \frac{R}{r_k}\left[\frac{1}{(1-r_n-....-r_k)} - \frac{1}{(1-r_n-....-r_{k+1})}\right]$$

Therefore $$r_k W_{q(k)} = R\left[\frac{1}{(1-r_n-....-r_k)} - \frac{1}{(1-r_n-....-r_{k+1})}\right]$$

We can now write the sum $\sum_{k=1}^{n} r_k W_{q(k)}$ as

$$\begin{aligned}
\sum_{k=1}^{n} r_k W_{q(k)} &= \frac{Rr_n}{1-r_n} + \sum_{k=1}^{n-1} r_k W_{q(k)} \\
&= \frac{Rr_n}{1-r_n} + R\left[\frac{1}{(1-r_n-....-r_1)} - \frac{1}{(1-r_n-....-r_2)}\right] \\
&\qquad + R\left[\frac{1}{(1-r_n-....-r_2)} - \frac{1}{(1-r_n-....-r_3)}\right] \\
&\qquad + ... \\
&\qquad + R\left[\frac{1}{(1-r_n-r_{n-1})} - \frac{1}{(1-r_n)}\right] \\
&= \frac{Rr_n}{1-r_n} + R\left[\frac{1}{(1-r)} - \frac{1}{(1-r_n)}\right] \qquad r = r_1 + ...+ r_n
\end{aligned}$$

Therefore

$$\sum_{k=1}^{n} r_k W_{q(k)} = \frac{R}{(1-r)} - R = \frac{Rr}{(1-r)}$$

Q.E.D.

An interesting alternate approach may also be considered. Assume that the queue is examined at an arbitrary time instant and let *R* be the mean residual service time observed at that time with *U* as the mean unfinished work in the queue. We can then write that

$$U = R + \sum_{k=1}^{n} N_{q(k)} \overline{X_k}$$

Applying Little's result individually for each priority class gives

$$U = R + \sum_{k=1}^{n} l_k W_{q(k)} \overline{X_k}$$

$$= R + \sum_{k=1}^{n} r_k W_{q(k)}$$

Since both *U* and *R* are not dependent on the priority order of the classes, $\sum_{k=1}^{n} r_k W_{q(k)}$ will also be independent of this. Moreover, for a single class queue, we can write that

$$U = R + r \frac{R}{1-r} = \frac{R}{1-r}$$

Therefore, we get that for *n* priority classes as well $U = \frac{R}{1-r}$. Using this, we get

$$\sum_{k=1}^{n} r_k W_{q(k)} = U - R = \frac{Rr}{1-r}$$ Q.E.D.