2.9 The Method of Stages for Solving a M/-/1/∞ FCFS Queue

This method is useful to analyse queues where the service time is not exponentially distributed but may be represented (at least approximately) as a sum of independent, exponentially distributed random variables. These variables may not be identical but should be independent. (As we will show later, other more complex cases may also be handled by this method.

To illustrate this method, consider a single server queue where the service time is a random variable consisting of the sum of two, independent, exponentially distributed random variables with means $1/m_1$ and $1/m_2$. The arrival process is Poisson with rate l and we assume that the queue has an infinite buffer. The operation of this queue may be illustrated as in Figure 2.7. In this queue, a customer starting service, first enters *Stage 1*, where it gets served for an exponentially distributed random time with mean $1/m_1$. On completion of this, it enters *Stage 2* for an exponentially distributed random time with mean $1/m_2$ and then departs from the system. New customers enter *Stage 1* only after the previous customer has left the system, i.e. left *Stage 2*. The overall service time would thus be the sum of the random service times at the two stages.

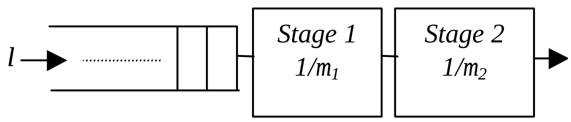

Figure 2.7. Single Sever Queue with Two Stages of Service

We can denote the state of the system in Figure 2.7 as *(n, j)* where *n* is the total number of customers in the system where the customer currently being served is at Stage *j*, *n=0,1,......,¥*, *j=1, 2*. We use *(0,0)* to denote the special case when the system is empty. The State Transition Diagram of this system will be as shown in Figure 2.8

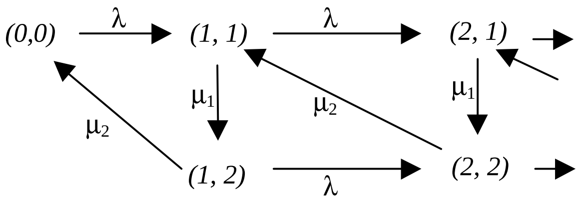

Figure 2.8. State Transition Diagram for Single Sever Queue with Two Stages of Service

The balance equations for this may be written as follows

$$
\begin{aligned}
&l p_{00} = m_2 p_{12} \\
&(l + m_1) p_{11} = l p_{00} + m_2 p_{22} \\
&(l + m_2) p_{12} = m_1 p_{11} \\
&(l + m_1) p_{21} = l p_{11} + m_2 p_{32} \\
&(l + m_2) p_{22} = l p_{12} + m_1 p_{21} \\
&etc.......
\end{aligned}
\tag{2.38}
$$

These may then be solved to get solutions for the state probabilities as a function of p_{00}, the probability of the system being empty, as

$$
\begin{aligned}
&p_{12} = \frac{l}{m_2} p_{00} \\
&p_{11} = \frac{l(l + m_2)}{m_1 m_2} p_{00} \\
&p_{22} = \left(\frac{l + m_1}{m_2} \right) p_{11} - \frac{l}{m_2} p_{00} \\
&etc......
\end{aligned}
\tag{2.39}
$$

The complete solution for the various state probabilities may then be found by using the normalisation condition of Eq. (2.4) to first find p_{00} and then the various p_{nj} $n=1,2,......,¥$, $j=1, 2$. This approach may be easily extended to allow for the following -

(1) Have k stages of service times - For this, we may extend the approach given above to allow *k* stages of service.

(2) Finite Waiting Positions in the Queue - This may be handled by making the arrival rate a function of the number in the system, i.e. make it go to zero when all the waiting positions have been filled.

(3) Multiple Servers - Approximation for this may be done by allowing more than one customer to enter service at a time.

(4) More General Service Distributions - A similar approach may be used to handle more general service time distributions whose Laplace Transforms (of their pdf's) may be represented as a rational function in s. These will be functions of the type $L_B(s)=N(s)/D(s)$ *with simple roots*. For this, consider the system with several stages of service, organised as shown in Figure 2.9

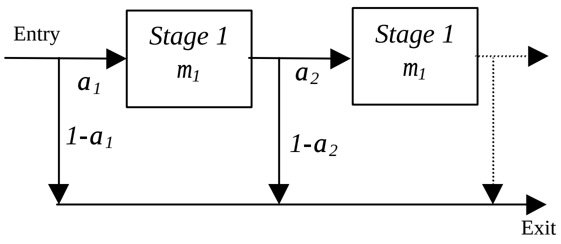

Figure 2.9. Generalised Service in Stages of a Single Server Queue

Consider a general system of the type shown in Figure 2.9 where the individual stages have independent exponentially distributed service times. We assume that the time spent in Stage *j* is an exponentially distributed random variable with mean $1/m_j$ and Laplace Transform $m_j/(s+m_j)$. The Laplace Transform $L_B(s)$ of the overall service may then be represented as

$$L_B(s) = (1-a_1) + \sum_{j\geq 2} a_1 \ldots\ldots a_{j-1}(1-a_j)\prod_{i=1}^{j}\frac{m_i}{s+m_i} \qquad (2.40)$$

This $L_B(s)$ has simple roots and may be expanded using partial fraction expansion in the following form.

$$L_B(s) = b_0 + \sum_i \frac{b_i}{s + m_i} \tag{2.41}$$

Note that the coefficients b_i $i=0, 1, 2, ...$ may be expressed as functions of the variables m_i and a_i.

Given a $L_B(s)$ that is a rational function of s with simple roots, we would need to first express it in the form given in Eq. (2.40). This may then be expanded in the form of partial fractions to obtain the form of Eq. (2.41) from which a model using stages may be built as shown in Figure 2.9. This model with service in stages may be analysed in the same way in which we had earlier illustrated the analysis where only two stages were involved. This would involve drawing its state transition diagram with a proper definition of the system state. This state transition diagram may then be used to obtain the corresponding balance equations. These can then be solved to obtain the equilibrium state probabilities of the system. Once the system state probabilities are known, other system parameters may be calculated as desired.