EC 633, Queueing Systems
Home Assignment No. 6
Solutions

1. Note that, as derived in class, the mean queueing delay W_{qk} of customers of priority class k will be given by

$$W_{qk} = \frac{R}{(1-\rho_n - .... - \rho_{k+1})(1-\rho_n - .... - \rho_k)}$$

This may be rewritten as

$$W_{qk} = \frac{R}{\rho_k}\left[\frac{1}{(1-\rho_n - .... - \rho_k)} - \frac{1}{(1-\rho_n - .... - \rho_{k+1})}\right]$$

Therefore $\rho_k W_{qk} = R\left[\frac{1}{(1-\rho_n - .... - \rho_k)} - \frac{1}{(1-\rho_n - .... - \rho_{k+1})}\right]$

We can now write the sum $\sum_{k=1}^{n} \rho_k W_{qk}$ as

$$\sum_{k=1}^{n} \rho_k W_{qk} = \frac{R\rho_n}{1-\rho_n} + \sum_{k=1}^{n-1} \rho_k W_{qk}$$

$$= \frac{R\rho_n}{1-\rho_n} + R\left[\frac{1}{(1-\rho_n - .... - \rho_1)} - \frac{1}{(1-\rho_n - .... - \rho_2)}\right]$$

$$+ R\left[\frac{1}{(1-\rho_n - .... - \rho_2)} - \frac{1}{(1-\rho_n - .... - \rho_3)}\right]$$

$$+ ....$$

$$+ R\left[\frac{1}{(1-\rho_n - \rho_{n-1})} - \frac{1}{(1-\rho_n)}\right]$$

$$= \frac{R\rho_n}{1-\rho_n} + R\left[\frac{1}{(1-\rho)} - \frac{1}{(1-\rho_n)}\right] \qquad \rho = \rho_1 + ... + \rho_n$$

Therefore $\sum_{k=1}^{n} \rho_k W_{qk} = \frac{R}{(1-\rho)} - R = \frac{R\rho}{(1-\rho)}$

Alternative Solution

An interesting alternate approach may also be considered. Assume that the queue is examined at an arbitrary time instant and let R be the mean residual service time observed at that time with U as the mean unfinished work in the queue. We can then write that

$$U = R + \sum_{k=1}^{n} N_{qk}\overline{X_k}$$

Applying Little's result individually for each priority class gives -

$$U = R + \sum_{k=1}^{n} \lambda_k W_{qk} \overline{X_k}$$

$$= R + \sum_{k=1}^{n} \rho_k W_{qk}$$

Since neither *U* nor *R* depend on the priority order of the classes, $\sum_{k=1}^{n} \rho_k W_{qk}$ will also be independent of this. Moreover, for a single class queue, we can write that

$$U = R + \rho \frac{R}{1-\rho} = \frac{R}{1-\rho}$$

Therefore, we get that for *n* priority classes as well we will have $U = \frac{R}{1-\rho}$.

Using this, we get $\sum_{k=1}^{n} \rho_k W_{qk} = U - R = \frac{R\rho}{1-\rho}$

2. LST of Class 2 busy period duration = $\tilde{F}_{B_2}(s) = e^{-sX_2}$

$$E\{e^{-sT} \mid u = X_1, n\} = e^{-s(u+nX_2)}$$

$$E\{e^{-sT} \mid u = X_1\} = e^{-su} \sum_{n=0}^{\infty} e^{-snX_2} \frac{(\lambda_2 u)^n}{n!} e^{-\lambda_2 u} = e^{-u\left(s+\lambda_2-\lambda_2 \exp(-sX_2)\right)}$$

Therefore, $\tilde{F}_T(s) = \exp\left(-X_1(s + \lambda_2 - \lambda_2 e^{-sX_2}\right)$

3. See lecture notes