EC 633, Queueing Systems
Home Assignment No. 5
Solutions

For solutions to Problems 1 and 2, refer to lecture notes and class discussions

3.

(a) Let the *Effective Service Time Distribution* (Laplace Transform) of a job entering the system (from outside) be $L_{B^*}(s)$.

Therefore $L_{B^*}(s) = \sum_{k=1}^{\infty} pL_B(s)[L_B(s)]^{k-1} = \frac{pL_B(s)}{1-qL_B(s)}$

with Mean Effective Service Time $\overline{X^*} = \sum_{k=1}^{\infty} pq^{k-1}(k\overline{X}) = \frac{p\overline{X}}{(1-q)^2} = \frac{\overline{X}}{p}$

where $\overline{X}$ is the mean service time in the actual queue.

Effective Traffic = $\rho^* = \lambda\overline{X^*} = \frac{\lambda\overline{X}}{p} = \frac{\rho}{p}$ where $\rho = \lambda\overline{X}$

Following the analysis of an M/G/1 queue (and considering the whole system as a M/G/1 queue with the effective service time distribution $L_{B^*}(s)$ as given earlier), we get –

$$P(z) = p_0 \frac{(1-z)L_{B^*}(\lambda-\lambda z)}{L_{B^*}(\lambda-\lambda z)-z} \qquad \text{with } p_0 = 1-\rho^*$$

Simplifying this gives,

$$P(z) = (p-\rho)\frac{(1-z)L_B(\lambda-\lambda z)}{(p+qz)L_B(\lambda-\lambda z)-z}$$

Note that this is the generating function of the number *n* in the system that a customer departing the queue will see looking back immediately after it has left. However, with probability *p* the job will be immediately fed back to the queue. Therefore, the generating function at A will be

$$P_A(z) = (p+qz)P(z) = (p-\rho)\frac{(1-z)(p+qz)L_B(\lambda-\lambda z)}{(p+qz)L_B(\lambda-\lambda z)-z}$$

(b) For point B, the Markov chain may be written at the departure instants with n_i as the number seen left behind in the system by the i^{th} departure.

For $n_i=0$	$n_{i+1}= a_{i+1}$	probability = p
	$= a_{i+1}+1$	probability = q
For $n_i\geq 1$	$n_{i+1}= n_i+ a_{i+1}-1$	probability = p
	$= n_i+ a_{i+1}$	probability = q

Therefore, (with $A(z)=E[z^a]$, note that $A(z)=L_B(\lambda- \lambda z)$)

$$P_B(z) = A(z)p_{B0}[p+qz] + A(z)pz^{-1}[P_B(z) - p_{B0}] + A(z)q[P_B(z) - p_{B0}]$$
$$P_B(z) = p_{B0}\frac{(1-z)(p+qz)A(z)}{[(p+qz)A(z)-z]}$$

Directly taking means of the LHS and RHS of the Markov Chain expressions at equilibrium and using $E\{n_i\} = E\{n_{i+1}\} = N$ and $E\{a_{i+1}\} = \lambda\overline{X} = \rho$, we get

$$N = N + \lambda\overline{X} + p_{B0}q - (1-p_{B0})p \qquad \Rightarrow \qquad p_{B0} = p - \rho$$
$$0 = \rho + p_{B0} - p$$

Therefore $P_B(z) = (p-\rho)\frac{(1-z)(p+qz)A(z)}{[(p+qz)A(z)-z]} = (p-\rho)\frac{(1-z)(p+qz)L_B(\lambda-\lambda z)}{[(p+qz)L_B(\lambda-\lambda z)-z]}$

4. To prove $A_k = \int_0^\infty \frac{(\lambda x)^{k-1}}{(k-1)!}e^{-\lambda x}[1-B(x)]\lambda dx \qquad k=1,2,\ldots\ldots..\infty$ **(A)**

Note that since $A_k = A_{k+1}$ + P{*k* arrivals in the time interval}

$$A_{k+1} = A_k - \int_0^\infty \frac{(\lambda x)^k}{k!}e^{-\lambda x}b(x)dx \qquad \textbf{(B)}$$

We prove **(A)** by mathematical induction by first showing that it holds for *k*=1 and then using the recursion of **(B)** to show that if it holds for *k* then it will also hold for *k*+1

For *k*=1

$$A_1 = \sum_{j=1}^\infty \int_0^\infty \frac{(\lambda x)^j}{j!}e^{-\lambda x}b(x)dx = \int_0^\infty (1-e^{-\lambda x})b(x)dx = 1 - \int_0^\infty e^{-\lambda x}b(x)dx$$
$$= \lambda\int_0^\infty e^{-\lambda x}dx - \left[e^{-\lambda x}B(x)\Big|_0^\infty + \lambda\int_0^\infty e^{-\lambda x}B(x)dx\right]$$
$$= \int_0^\infty e^{-\lambda x}[1-B(x)]\lambda dx$$

Using the recursion for **(B)** and assuming **(A)** holds for *k*, we get the following for *k*+1.

$$A_{k+1} = \int_0^\infty \frac{(\lambda x)^{k-1}}{(k-1)!}e^{-\lambda x}[1-B(x)]\lambda dx - \int_0^\infty \frac{(\lambda x)^k}{k!}e^{-\lambda x}b(x)dx \qquad \textbf{(C)}$$

Integrating by parts, we can show that

$$\int_0^\infty \frac{(\lambda x)^{k-1}}{(k-1)!}e^{-\lambda x}[1-B(x)]\lambda dx = \int_0^\infty \frac{(\lambda x)^k}{k!}e^{-\lambda x}\left[\lambda(1-B(x)) + b(x)\right]dx \qquad \textbf{(D)}$$

Substituting **(D)** in **(C)** gives the desired result.