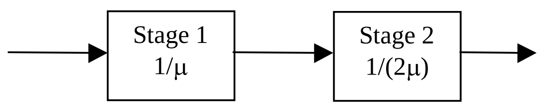
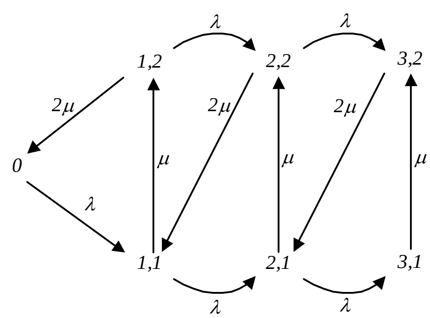
EC 633, Queueing Systems
Home Assignment No. 3
Solutions

1. The system's server may be represented by the service facility shown. Here stages 1 and 2 serve with exponentially distributed service times with means μ^{-1} and $(2\mu)^{-1}$, respectively.



As usual, we represent the system state by (*n, m*) where *n* is the number in the system and *m* is the stage at which the currently served customer may be found. The corresponding state transition diagram is given below.



The balance equations may be written as follows.

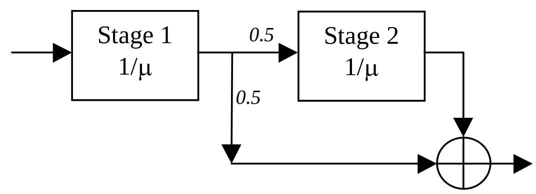
$$\lambda p_0 = 2\mu p_{12} \qquad (\lambda + 2\mu) p_{12} = \mu p_{11} \qquad \lambda p_{21} = \mu p_{31}$$
$$(\lambda + \mu) p_{21} = \lambda p_{11} + 2\mu p_{32} \qquad 2\mu p_{32} = \lambda p_{22} + \mu p_{31}$$
$$(\lambda + \mu) p_{11} = \lambda p_0 + 2\mu p_{22} \qquad (\lambda + 2\mu) p_{22} = \lambda p_{12} + \mu p_{21}$$

These may be solved for the state probabilities from which the individual state probabilities may be found as

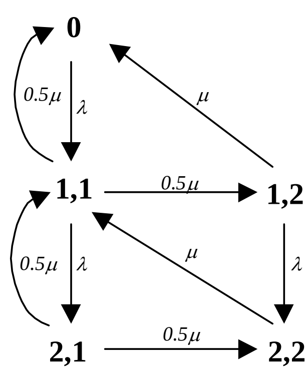
$$P\{n = 0\} = p_0 \qquad P\{n = 1\} = p_{11} + p_{12}$$
$$P\{n = 2\} = p_{21} + p_{22} \qquad P\{n = 3\} = p_{31} + p_{32}$$

The probability that an arrival is blocked and leaves without service is the same as finding the system in state 3 with probability ($p_{31} + p_{32}$).

2. The system's server may be represented by the service facility shown below.



As usual, we represent the system state by (*n, m*) where *n* is the number in the system and *m* is the stage at which the currently served customer may be found. The corresponding state transition diagram is given below.



The corresponding balance equations are

$$\lambda p_0 = 0.5\mu p_{11} + \mu p_{12} \qquad (\lambda + \mu) p_{11} = \lambda p_0 + 0.5\mu p_{21} + \mu p_{22}$$
$$(\lambda + \mu) p_{12} = 0.5\mu p_{11} \qquad \mu p_{21} = \lambda p_{11} \qquad \mu p_{22} = \lambda p_{12} + 0.5\mu p_{21}$$

Choosing any four of these (i.e. the most convenient ones), we can get the state probabilities as

$$p_{11} = p_0 \frac{2\rho(1+\rho)}{2+\rho} \qquad p_{12} = p_0 \frac{\rho}{2+\rho}$$
$$p_{21} = p_0 \frac{2\rho^2(1+\rho)}{2+\rho} \qquad p_{22} = p_0 \rho^2$$

Applying the normalisation condition to these, we get

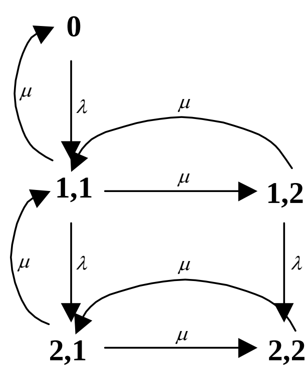
$$p_0 = \frac{2+\rho}{2+4\rho+6\rho^2+3\rho^3}$$

The *Average Departure Rate from the Queue* may be found using either of the two approaches -

(a) $0.5\mu(p_{11}+p_{21})+\mu(p_{12}+p_{22})$ or (b) $\lambda(p_0+p_{11}+p_{12})$

Using either approach, we get this to be $\lambda\left(2p_0\frac{(1+\rho)^2}{(2+\rho)}\right)$

3. The state transition diagram for this system will be as shown below. The state is represented as (*m, j*) where *m* is the number of jobs in the system and *j* is the stage that the job is currently in.



Choosing convenient closed boundaries, the balance equations for this system may be written as

$$\lambda p_0 = \mu p_{11} = (\lambda+\mu)p_{12}$$
$$\lambda(p_{11}+p_{12}) = \mu p_{21}$$
$$p_{22}+p_{12} = p_{11}+p_{21}$$

which gives

$$p_{11} = \rho p_0 \quad p_{12} = \frac{\rho}{1+\rho}p_0 \quad p_{21} = \frac{\rho^2(2+\rho)}{1+\rho}p_0 \quad p_{22} = \frac{\rho^2(3+\rho)}{1+\rho}p_0$$

Applying the normalisation condition to this, we get

$$p_0 = \frac{1+\rho}{1+3\rho+6\rho^2+2\rho^3}$$

The probability P_B that an arrival leaves without service is (p_{21}+p_{22}). This leads to

$$P_B = \frac{\rho^2(5+2\rho)}{(1+\rho)}p_0 \qquad \text{and} \qquad \lambda_{eff} = \lambda(1-P_B) = \lambda\frac{1+3\rho+\rho^2}{1+3\rho+6\rho^2+2\rho^3}$$

Since $N_q = p_{21}+p_{22} = \frac{\rho^2(5+2\rho)}{1+3\rho+6\rho^2+2\rho^3}$, we get $W_q = \frac{\rho^2(5+2\rho)}{\lambda(1+3\rho+\rho^2)}$

The Laplace Transform of the effective overall service distribution will be given by

$$L_B(s) = 0.5\left(\frac{2\mu}{s+2\mu}\right)\sum_{i=0}^{\infty}\left[0.5\left(\frac{2\mu}{s+2\mu}\right)\left(\frac{\mu}{s+\mu}\right)\right]^i$$
$$= \frac{\mu(s+\mu)}{(s^2+3\mu s+\mu^2)}$$