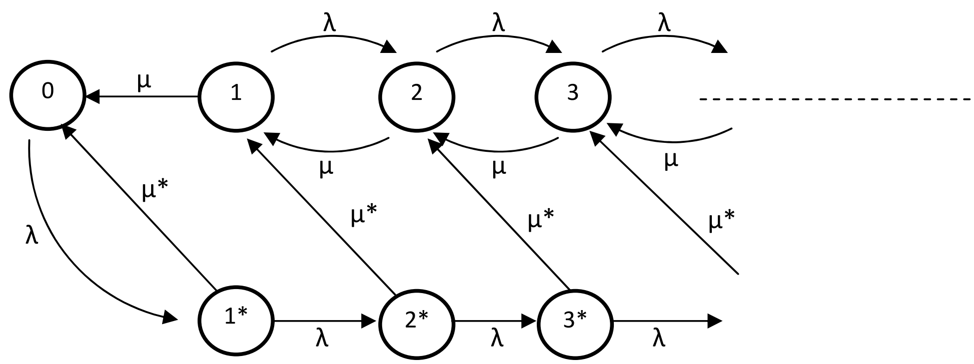
EC 633, Queueing Systems
Home Assignment No. 2
Solutions

1. The State Transition Diagram for this queue may be drawn as follows –



Here μ* is the service rate at which the first customer after the queue goes idle is served. The state *n** represents the case where there are *n* customers in the system and the customer being served is the one whose service started first after the queue became idle.

We can now write the balance equations that need to be solved (along with the normalization condition) to obtain the state probabilities.

$$p_1\mu + p_{1*}\mu^* = p_0\lambda$$
$$p_2\mu + p_{2*}\mu^* = p_1(\lambda+\mu) \qquad p_{1*}(\lambda+\mu^*) = p_0\lambda$$
$$p_3\mu + p_{3*}\mu^* = p_2(\lambda+\mu) - p_1\lambda \qquad p_{2*}(\lambda+\mu^*) = p_{1*}\lambda$$
$$p_4\mu + p_{4*}\mu^* = p_3(\lambda+\mu) - p_2\lambda \qquad p_{3*}(\lambda+\mu^*) = p_{2*}\lambda$$

....

.... $$p_{n*} = \left(\frac{\lambda}{\lambda+\mu}\right)^n p_0$$

Normalization Condition: $$p_0 + \sum_{n=1}^{\infty}(p_n + p_{n*}) = 1$$

Though the individual probabilities can be found from the above, the expressions do get messy. Fortunately, p_0 can be directly evaluated as follows.

Consider a cycle which starts when the queue becomes empty for the j^{th} time and ends continues until the queue once again becomes empty for the $(j+1)^{th}$ time. Let T_{cycle} be the average cycle time consisting of an idle period of average length λ^{-1} and a busy period of average length $\overline{BP}$. Assume $\overline{X+\Delta} = \overline{X} + \overline{\Delta}$ to be the mean service time for the first customer in the busy period and let $\overline{X} = \mu^{-1}$ be the mean service time of the other customers in the busy period. Then, we can show using our earlier arguments that

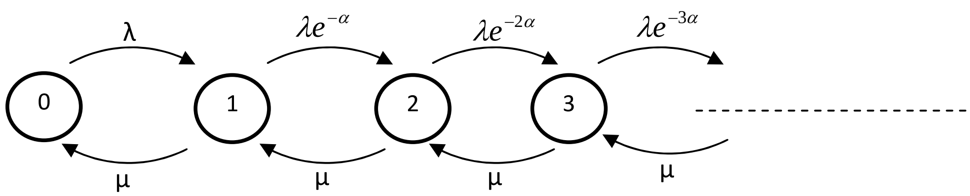
$$\overline{BP} = \overline{X} + \overline{\Delta} + \lambda\left(\overline{X} + \overline{\Delta}\right)\frac{\overline{X}}{1-\lambda\overline{X}} = \frac{\overline{X} + \overline{\Delta}}{1-\lambda\overline{X}}$$

Therefore,

$$p_0 = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{\overline{X} + \overline{\Delta}}{1-\lambda\overline{X}}} = \frac{1-\lambda\overline{X}}{1+\lambda\overline{\Delta}}$$

This can be directly substituted in the probability expressions obtained from the balance equations to obtain the individual state probabilities.

2. The State Transition Diagram for this queue may be drawn as follows –

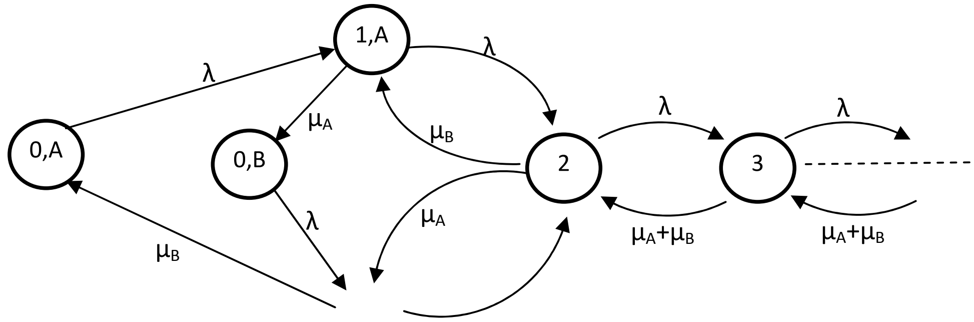


Applying flow balance, using $\rho = \frac{\lambda}{\mu}$, we can get that $p_n = \rho^n e^{-\frac{(n-1)n}{2}\alpha} p_0$ n=1, 2,....
where p_0 may once again be applied by using the normalization condition. (We cannot get a closed form expression for p_0 – at least, I can't!)

For $\alpha \to \infty$., this system degenerates to a system with only two states, 0 and 1, where there are no waiting positions and arrivals who find the server busy leave without service.

3. A State Transition Diagram for this system is given below. Note that states 2, 3,...∞ are normally defined. The other states are defined as follows –

{1,A} one customer in the system, Server A working
{1,B} one customer in the system, Server B working
{0,A} system empty, Server A idle for longer time than Server B
{0,B} system empty, Server B idle for longer time than Server A



Solving the balance equations for this we get –

$$p_{0A} = p_{0B} = \frac{\mu_A \mu_B}{\lambda^2} p_2 \qquad p_{1A} = \frac{\mu_B}{\lambda} p_2 \quad p_{1B} = \frac{\mu_A}{\lambda} p_2$$

$$p_n = \left(\frac{\lambda}{\mu_A + \mu_B} \right)^{n-2} p_2 \qquad n = 2,3,4,.......$$

The normalization condition will then give

$$p_2 = \frac{1}{\left(2\frac{\mu_A \mu_B}{\lambda^2} + \frac{\mu_A + \mu_B}{\lambda} + \frac{\mu_A + \mu_B}{\mu_A + \mu_B - \lambda} \right)}$$

Using p_2 from above, the required probabilities in terms of p_2 are –

$$p_0 = 2\frac{\mu_A \mu_B}{\lambda^2} p_2$$

$$p_1 = \frac{\mu_A + \mu_B}{\lambda} p_2$$

$$p_n = \left(\frac{\lambda}{\mu_A + \mu_B} \right)^{n-2} p_2 \qquad n = 2,3,4,.......$$

4. We give two different ways in which the problem may be done as both illustrate interesting ways for analyzing the system (without doing a brute force solution)!

   Method 1: For Thursdays and Fridays, the time axis of the system may be viewed as alternating periods when the server is working and is on vacation. In one such cycle, let T_{TV} be the average total vacation period (in a cycle) which will consist of one or more vacation intervals (each of mean length β^{-1}) and let T_B be the average total time in a cycle during which the server is busy. We can then find P{server idle}=$T_{TV}/(T_{TV} + T_B)$ as described below.

   When there are no arrivals in a vacation interval (probability $\frac{\beta}{(\lambda+\beta)}$), the server goes on another vacation. From this, we can derive that the average number of vacations before a busy period starts once again are $\frac{(\lambda+\beta)}{\lambda}$ and also show that –

   P{ n arrivals in a vacation interval}

$$= \int_0^\infty \frac{(\lambda v)^n}{n!} e^{-\lambda v} \beta e^{-\beta v} dv = \frac{\beta}{(\lambda+\beta)} \left(\frac{\lambda}{\lambda+\beta} \right)^n \qquad n \geq 0$$

   P {n arrivals in a vacation interval | vacation has at least one arrival}

$$= \frac{\frac{\beta}{(\lambda+\beta)}\left(\frac{\lambda}{\lambda+\beta}\right)^n}{1-\frac{\beta}{(\lambda+\beta)}} = \frac{\beta}{\lambda}\left(\frac{\lambda}{\lambda+\beta}\right)^n \qquad n\geq 1$$

Mean number of arrivals in the last vacation interval of a vacation period $= \frac{\lambda+\beta}{\beta}$

Therefore $T_{TV} = \frac{\lambda+\beta}{\lambda\beta} = \frac{1}{\lambda} + \frac{1}{\beta}$

Recall from the lectures (or derive as was done there) that the mean length of a busy period in a normal M/M/1 queue is $\frac{\mu^{-1}}{1-\rho}$. Therefore, for this system

Mean length of the Busy Period = $\frac{\mu^{-1}}{1-\rho}$(Mean number of arrivals in the last vacation interval of a vacation period).

Therefore, $T_B = \left(\frac{1}{\lambda} + \frac{1}{\beta}\right)\frac{\rho}{1-\rho}$

Using these values of T_B and T_{TV}, we get that for Thursdays and Fridays,

P{server idle} = 1-ρ

P{server busy}=1-P{server idle}=ρ

For Monday-Wednesday, the system is a normal M/M/1 queue with the same probability of server being busy (i.e. ρ). So the clerk works the same amount on all days and should not be given extra payment for Monday-Wednesday.

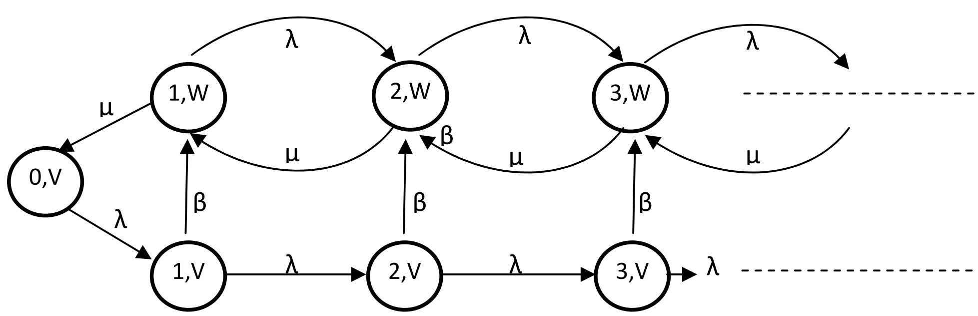
Method 2: We can draw a state transition diagram for the way the server operates on Thursdays and Fridays using the following definition of the system state.

{*j*, W} Server working with *j* customers in the system *j*=1,2,3,....∞

{*k*, V} Server on vacation with *k* customers in the system *k*=0,1,2,3,....∞

Note that the system is empty only when it is in state {0, V} but that the server is idle in all states {*k*, V} for $\forall k$

The state transition diagram is given below –



We can write the balance equations along with the normalization condition to find the probabilities of each of the system states and then calculate the probability of the server being idle (i.e. on vacation) as $\sum_{k=0}^{\infty} p_{kV}$. However, just to answer Miss Bannerjee's question, we do not really need to do all that!

Using the balance conditions and some simple manipulations, we can get the following –

$$p_{nV} = \left(\frac{\lambda}{\lambda+\beta}\right)^n p_{0V} \quad n = 0,1,2,\ldots\ldots\infty \qquad \Rightarrow \sum_{n=0}^{\infty} p_{nV} = \frac{\lambda+\beta}{\beta} p_{0V}$$

and

$$\begin{aligned} p_{1W} &= \rho p_{0V} & n = 1 \\ p_{nW} &= \rho p_{(n-1)W} + \rho p_{(n-1)V} & n = 2,3,\ldots\ldots\infty \end{aligned}$$

Summing the LHS and RHS of the above from *n*=1 to *n*=∞, we get

$$\sum_{n=1}^{\infty} p_{nW} = \rho \sum_{n=1}^{\infty} p_{nW} + \rho \frac{\lambda+\beta}{\beta} p_{0V}$$

$$(1-\rho)\sum_{n=1}^{\infty} p_{nW} = \rho \frac{\lambda+\beta}{\beta} p_{0V} \qquad \Rightarrow \sum_{n=1}^{\infty} p_{nW} = \frac{\rho}{1-\rho}\frac{\lambda+\beta}{\beta} p_{0V}$$

Summing all states, we get –

$$1 = \sum_{n=0}^{\infty} p_{nV} + \sum_{n=1}^{\infty} p_{nW} = \left(\frac{\lambda+\beta}{\beta}\right) p_{0V} \left(\frac{1}{1-\rho}\right) \qquad \Rightarrow p_{0V} = \frac{\beta(1-\rho)}{(\lambda+\beta)}$$

Therefore $\sum_{n=0}^{\infty} p_{nV} = (1-\rho)$ implying that the probability of the server working remains as ρ. Hence, as claimed earlier, there is no reason for Miss Bannerjee to pay the clerk more for the work done on Monday-Wednesday.