EC 633, Queueing Systems
Home Assignment No. 1
Solutions

1. Let $p_k(t)$ = P{k particles arrive till time t} k=0,1,2,.......

   Then

$$p_0(t+\Delta t) = p_0(t)(1-\lambda\Delta t)$$
$$p_k(t+\Delta t) = p_k(t)(1-\lambda\Delta t) + p_{k-1}(t)\lambda\Delta t \qquad k = 1,2,.....$$

   Taking limits as $\Delta t \to 0$, we get

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t)$$
$$\frac{dp_k(t)}{dt} = -\lambda\left[p_k(t) - p_{k-1}(t)\right] \qquad k = 1,2,.....$$

   Solving these for $p_k(t)$ with the initial condition $p_0(0)$=1 will give us the solution

$$p_k(t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \qquad k = 0,1,2,.......$$

   A convenient way to solve this will be by using transforms intelligently rather than solving painfully for each $p_i(t)$. For this, multiply the i^{th} equation by z^i and sum to get

$$\frac{d}{dt}\left(\sum_{k=0}^{\infty} z^k p_k(t)\right) = -\lambda\left(\sum_{k=0}^{\infty} z^k p_k(t)\right) + \lambda\left(\sum_{k=1}^{\infty} z^k p_{k-1}(t)\right)$$

   Defining $P(z,t) = \sum_{k=0}^{\infty} z^k p_k(t)$ as the z-transform of $p_k(t)$, the above equation may be written as $\quad \frac{dP(z,t)}{dt} = -\lambda P(z,t) + \lambda z P(z,t) = -\lambda(1-z)P(z,t)$

   Note that the initial condition given is that the counter is in state 0 at time t=0. This implies that $P(z,0)=1$ which may be used as the initial condition for the differential equation given above for $P(z,t)$. Also note that the normalization condition $\sum_{k=0}^{\infty} p_k(t) = 1$ implies that $P(1,t)=1$ for all t

   The differential equation for $P(z,t)$ can be easily solved.

$$\frac{dP(z,t)}{P(z,t)} = -\lambda(1-z)dt \qquad \Rightarrow \quad \ln P(z,t) = -\lambda(1-z)t + K$$
$$P(z,t) = Ce^{-\lambda(1-z)t} \qquad \text{but } P(z,0)=1 \qquad \Rightarrow \quad C=1$$

   Therefore $P(z,t) = e^{-\lambda(1-z)t}$

   This may be inverted to give $p_k(t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \qquad k = 0,1,2,.......$

N.B.: *This was an example of a Pure-Birth process as we had only arrivals but no departures. We can similarly have a Pure-Death process where the system starts from a known initial state (which is non-zero) and has only departures until the system reduces to zero! Consider analyzing that with suitable choices of parameters. Here one may consider two different scenarios – one where the departure rate is constant and another where the departure rate is proportional to the system state. (Even for the Pure Birth process, we could have assumed that the arrival rate varies depending on the state of the system.)*

2.

(a) Starting from a time instant when both machines are working, let t_{B1} and t_{B2} be the time to break-down of MC1 and MC2, respectively.

$$P\{t_{B2} > t_{B1} \mid t_{B1} = x\} = \int_x^\infty \mu_2 e^{-\mu_2 t} dt = e^{-\mu_2 x}$$

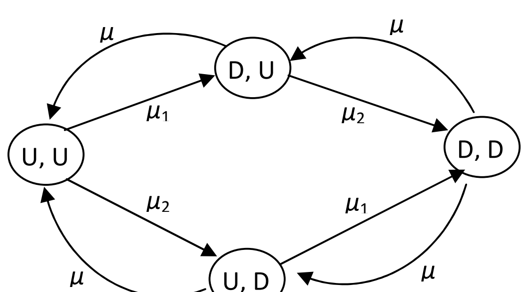
$$P\{t_{B2} > t_{B1}\} = \int_0^\infty \mu_1 e^{-\mu_1 x} e^{-\mu_2 x} dx = \frac{\mu_1}{\mu_1 + \mu_2}$$

(b) Starting from a time when MC1 is down but MC2 is up, let t_{U1} be the time to repair MC1 and t_{B2} be the time for MC2 to break down. Then

$$P\{t_{U1} > t_{B2} \mid t_{B2} = x\} = \int_x^\infty \mu e^{-\mu t} dt = e^{-\mu x}$$

$$P\{t_{U1} > t_{B2}\} = \int_0^\infty \mu_2 e^{-\mu_2 x} e^{-\mu x} dx = \frac{\mu_2}{\mu_2 + \mu}$$

(c) The state transition diagram for the system can be drawn as follows where U represents a machine being up and D represents the machine being down for repairs.



Write flow balance equations for this along with the normalization condition to get the following equilibrium probabilities –

$$P_{UU} = \frac{\mu^2}{\left(\mu^2 + \mu(\mu_1 + \mu_2) + \mu_1\mu_2\right)}$$

$$P_{DU} = \frac{\mu_1}{\mu} P_{UU} \qquad P_{UD} = \frac{\mu_2}{\mu} P_{UU} \qquad P_{DD} = \frac{\mu_1\mu_2}{\mu^2} P_{UU}$$

(i) P{both machines are working} = P_{UU}

(j) P{neither machine is working} = P_{DD}

(k) P{MC1 is working} = P_{UU} + P_{UD}

(l) P{MC2 is working} = $P_{UU} + P_{DU}$
(m) P{MC1 is working but MC2 is down} = P_{UD}
(n) P{MC2 is down but MC1 is working} = P_{DU}

3. Assume that the first student arrives at time t_1 and the second student arrives at time t_2 where $t_2 > t_1$. Moreover, since the arrivals come from a Poisson process, the random variable ($t_2 - t_1$) will have the pdf $f_{(t_2-t_1)}(\tau) = \lambda e^{-\lambda\tau}$ for $\tau \geq 0$

   (a) **For $X=c$ (constant)**
   The second student who arrives does not have to wait if $\tau \geq c$. Therefore –

   P{second student does not wait}= $\int_c^\infty \lambda e^{-\lambda\tau} d\tau = e^{-\lambda c}$

   Average waiting time for the second student

   $$= \int_0^c (c-\tau)\lambda e^{-\lambda\tau} d\tau = (c - \frac{1}{\lambda}) + \frac{1}{\lambda} e^{-\lambda c}$$

   (b) **For X is exponentially distributed with mean μ^{-1}**
   In this case, X is exponentially distributed with parameter μ. Therefore

   P{second student does not wait|$X=x$ }= $\int_x^\infty \lambda e^{-\lambda\tau} d\tau = e^{-\lambda x}$

   P{second student does not wait} = $\int_0^\infty e^{-\lambda x} \mu e^{-\mu x} dx = \frac{\mu}{\lambda + \mu}$

   Average waiting time for the second student

   $$= \int_0^\infty \left((x - \frac{1}{\lambda}) + \frac{1}{\lambda} e^{-\lambda x} \right) \mu e^{-\mu x} dx = \frac{1}{\mu} - \frac{1}{\lambda} + \frac{\mu}{\lambda(\mu+\lambda)} = \frac{\lambda}{\mu(\lambda+\mu)}$$

4. For this problem, note that when all k servers are busy the overall service rate will be $k\mu$. At that time, the time to the next departure will be exponentially distributed with mean $(k\mu)^{-1}$. This is really the key thing that should be kept in mind in this problem.
   (a) Let t_A be the time to the next arrival and let t_D be the time to the next departure. Then $\{t_A < t_D\}$ is the event of an arrival happening before a departure (and hence the arrival is forced to leave without service).

   $$P\{t_A < t_D \mid t_D = x\} = 1 - e^{-\lambda x}$$

   $$P\{t_A < t_D\} = \int_{x=0}^\infty (1 - e^{-\lambda x}) K\mu e^{-K\mu x} dx = 1 - \frac{K\mu}{\lambda + K\mu} = \frac{\lambda}{\lambda + K\mu}$$

   Good to check the solution for logical limits, i.e. when the arrival rate is high this probability should tend to 1 (it does!). Conversely, when servers serve fast, this probability should decrease!
   (b) Let t_A be the time to the next arrival, t_{D1} be the time to the first departure and t_{D2} be the time from the first departure to the next (second) departure. We really want to find the probability of the event $\{t_A > t_{D1} + t_{D2}\}$.

One way to do this would be to just extend the earlier approach as follows

$$P\{t_A > t_{D1} + t_{D2} \mid t_{D1} = x_1, t_{D2} = x_2\} = e^{-\lambda(x_1+x_2)}$$

$$P\{t_A > t_{D1} + t_{D2}\} = \int_{x_2=0}^{\infty} \left(e^{-\lambda x_2}(K-1)\mu e^{-(K-1)\mu x_2}\right) \int_{x_1=0}^{\infty} \left(e^{-\lambda x_1} K\mu e^{-K\mu x_1}\right) dx_1 dx_2$$

$$= \int_{x_2=0}^{\infty} \left(e^{-\lambda x_2}(K-1)\mu e^{-(K-1)\mu x_2}\right) \frac{K\mu}{\lambda + K\mu} dx_2 = \frac{K(K-1)\mu^2}{(\lambda + K\mu)(\lambda + (K-1)\mu)}$$

Alternatively, we can define a variable *Y* as the time to the second departure and find the pdf of *Y* first and then find the probability of the event $\{t_A > Y\}$. Let *X*1 be the time to the first departure and *X*2 be the time from the first departure to the second departure. Then

$$Y = X1 + X2$$

Note that the random variables *X*1 and *X*2 are independent. Therefore,

$$E\{e^{-sY}\} = E\{e^{-sX1}\} E\{e^{-sX2}\}$$

We know that $E\{e^{-sX1}\} = \dfrac{K\mu}{s + K\mu}$ and $E\{e^{-sX2}\} = \dfrac{(K-1)\mu}{s + (K-1)\mu}$

$$L_Y(s) = E\{e^{-sY}\} = \frac{K(K-1)\mu^2}{((s + K\mu)(s + (K-1)\mu))}$$

$$= K(K-1)\mu\left[\frac{1}{s + (K-1)\mu} - \frac{1}{s + K\mu}\right]$$

This will be the Laplace Transform of the pdf of *Y*. Inverting, we get

$$f_Y(y) = K(K-1)\mu\left(e^{-(K-1)\mu y} - e^{-K\mu y}\right)$$

Therefore the required probability would be –

$$\int_{y=0}^{\infty} K(K-1)\mu\left(e^{-(K-1)\mu y} - e^{-K\mu y}\right) e^{-\lambda y} dy = K(K-1)\mu\left(\frac{1}{\lambda + (K-1)\mu} + \frac{1}{\lambda + K\mu}\right)$$

which gives the same answer as before.

(c) Let *T* be the time to the first arrival after *t*=0. Then the average number of customers who will get turned away before any of the servers become free will be

$$\int_{T=0}^{\infty} \lambda T K\mu e^{-K\mu T} dT = \frac{\lambda}{K\mu}$$