EE 633, Quiz –II (30-MAR-2012)
Solutions

1. Let p_{di} = P{system in state *I* at the departure instant} *i*=0, 1
and $p_{d,jk}$= Probability of Transition from state *j* to state *k* from one departure instant to the next

(a) Transition Probabilities: $p_{d,00} = L_B(\lambda)$ $p_{d,01} = 1 - L_B(\lambda)$ $p_{d,10} = L_B(\lambda)e^{-\lambda\Delta}$ $p_{d,11} = 1 - L_B(\lambda)e^{-\lambda\Delta}$

Balance Equation: $p_{d,0} = p_{d,0} L_B(\lambda) + p_{d,1} L_B(\lambda)e^{-\lambda\Delta}$

Normalization Condition: $p_{d,0} + p_{d,1} = 1$

Solving, we get $p_{d,0} = \dfrac{L_B(\lambda)e^{-\lambda\Delta}}{1 - L_B(\lambda)\left[1 - e^{-\lambda\Delta}\right]}$ $p_{d,1} = \dfrac{1 - L_B(\lambda)}{1 - L_B(\lambda)\left[1 - e^{-\lambda\Delta}\right]}$

(b) Mean time between successive departures = $p_{d,0}\left(\dfrac{1}{\lambda} + \overline{X}\right) + p_{d,1}\left(\overline{X} + \Delta\right) = \overline{X} + \Delta p_{d,1} + \dfrac{1}{\lambda} p_{d,0}$

Mean idle time between successive departures = $\dfrac{1}{\lambda} p_{d,0}$

$$p_0 = \frac{\frac{1}{\lambda} p_{d,0}}{\overline{X} + \Delta p_{d,1} + \frac{1}{\lambda} p_{d,0}} = \frac{p_{d0}}{\lambda\overline{X} + \lambda\Delta(1 - p_{d,0}) + p_{d,0}} \text{ or } \frac{p_{d0}}{\lambda\overline{X} + \lambda\Delta p_{d,1} + p_{d,0}}$$

$$= \frac{p_{d,0}}{\lambda(\overline{X} + \Delta) + p_{d,0}(1 - \lambda\Delta)} = \frac{L_B(\lambda)e^{-\lambda\Delta}}{\lambda(\overline{X} + \Delta)\left(1 - L_B(\lambda)\left[1 - e^{-\lambda\Delta}\right]\right) + (1 - \lambda\Delta)L_B(\lambda)e^{-\lambda\Delta}}$$

$$= \frac{L_B(\lambda)e^{-\lambda\Delta}}{\lambda(\overline{X} + \Delta)[1 - L_B(\lambda)] + (1 + \lambda\overline{X})L_B(\lambda)e^{-\lambda\Delta}}$$

Alternative Approach: $p_0 = (1 - P_B)p_{do} = 1 - \hat{\rho}(1 - P_B)$ or $1 - P_B = \dfrac{1}{\hat{\rho} + p_{d0}}$

where, $\hat{\rho} = \lambda[\overline{X} + \Delta p_{d1}]$ which gives $1 - P_B = \dfrac{1}{\lambda\overline{X} + \lambda\Delta p_{d1} + p_{d0}}$

Therefore, $p_0 = \dfrac{p_{d0}}{\lambda\overline{X} + \lambda\Delta p_{d,1} + p_{d,0}}$

Another Alternative Approach: Consider the Mean Idle Period and Mean Busy Period in a Busy-Idle Cycle (using result from (c)

Mean Idle Period = $\overline{IP} = \dfrac{1}{\lambda}$

Mean Busy Period = $\overline{BP} = \dfrac{[1 - L_B(\lambda)](\overline{X} + \Delta) + \overline{X}L_B(\lambda)e^{-\lambda\Delta}}{L_B(\lambda)e^{-\lambda\Delta}}$

and use $p_0 = \dfrac{\overline{IP}}{\overline{IP} + \overline{BP}}$ to get the same result as before

(c) $\overline{BP} = \overline{X} + (\overline{X} + \Delta)(1 - p_{d,00})\, p_{d,10} \sum_{n=1}^{\infty} n(1 - p_{d,10})^{n-1}$

Therefore, $\overline{BP} = \overline{X} + (\overline{X} + \Delta)\dfrac{1 - p_{d,00}}{p_{d,10}} = \overline{X} + (\overline{X} + \Delta)\dfrac{1 - L_B(\lambda)}{L_B(\lambda)e^{-\lambda\Delta}}$

2. We consider each class separately, starting from the highest priority class

Class 3: Mean Residual Lifetime $R_3 = \dfrac{1}{2}\left(\lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)$

$$W_{q3} = R_3 + \overline{X}_3 N_{q3} \quad \Rightarrow \quad W_{q3} = \frac{R_3}{(1-\rho_3)} \quad \text{with } \rho_3 = \lambda_3 \overline{X}_3$$

$$W_3 = \overline{X}_3 + \frac{1}{2(1-\rho_3)}\left(\lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)$$

Class 2: Mean Residual Lifetime $R_2 = R_3 = \dfrac{1}{2}\left(\lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)$

$$W_{q2} = R_2 + N_{q3}\overline{X}_3 + \overline{X}_3 \lambda_3 W_{q2} + \overline{X}_2 N_{q2}$$

$$\Rightarrow \quad W_{q2} = \frac{R_2 + \rho_3 W_{q3}}{(1-\rho_2-\rho_3)} = \frac{R_2}{(1-\rho_3)(1-\rho_2-\rho_3)} \quad \text{with } \rho_2 = \lambda_2 \overline{X}_2$$

$$= \frac{\left(\lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)}{2(1-\rho_3)(1-\rho_2-\rho_3)}$$

$$W_2 = W_{q2} + \overline{X}_2 = \overline{X}_2 + \frac{\left(\lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)}{2(1-\rho_3)(1-\rho_2-\rho_3)}$$

Class 1: Mean Residual Lifetime $R_1 = \dfrac{1}{2}\left(\lambda_1 \overline{X_1^2} + \lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)$

$$W_1 = \overline{X}_1 + \frac{R_1}{(1-\rho_1-\rho_2-\rho_3)} + \overline{X}_3 \lambda_3 W_1 + \overline{X}_2 \lambda_2 W_1 \quad \text{with } \rho_1 = \lambda_1 \overline{X}_1$$

$$\Rightarrow \quad W_1 = \frac{\overline{X}_1(1-\rho_1-\rho_2-\rho_3) + R_1}{(1-\rho_2-\rho_3)(1-\rho_1-\rho_2-\rho_3)}$$