EC 633, Queueing Systems, QUIZ-II (Solutions)

1. Consider an M/G/1 system where the first and second moments of a normal service time are $\overline{X}$ and $\overline{X^2}$, respectively. However, the first and second moments of the *first service duration in a busy period* are $2\overline{X}$ and $\overline{X^2}$, respectively. *Use the Residual Life approach* to find the mean waiting time in queue W_q and the mean time spent in system W for a job arriving to the system under equilibrium conditions.

As for the usual M/G/1 queue, we can write $W_q = \dfrac{R}{(1-\lambda\overline{X})}$.

Average Length of a Busy Period = $2\overline{X} + (2\lambda\overline{X})\dfrac{\overline{X}}{1-\lambda\overline{X}} = \dfrac{2\overline{X}}{1-\lambda\overline{X}}$

Mean Cycle Time = $T_C = \dfrac{1}{\lambda} + \dfrac{2\overline{X}}{1-\lambda\overline{X}} = \dfrac{1+\lambda\overline{X}}{\lambda(1-\lambda\overline{X})}$

$$R = \frac{1}{2}\lambda\overline{X^2} \quad (obviously!)$$

Therefore, $$W_q = \frac{\lambda\overline{X^2}}{2(1-\lambda\overline{X})}$$

Mean Service Time = $(1-p_0)\overline{X} + p_0(2\overline{X}) = (1+p_0)\overline{X}$

where we can find p_0 as $p_0 = \dfrac{1/\lambda}{T_C} = \dfrac{1-\lambda\overline{X}}{1+\lambda\overline{X}}$

Therefore, Mean Service Time = $\dfrac{2\overline{X}}{(1+\lambda\overline{X})}$

Therefore, $$W = \frac{\lambda\overline{X^2}}{2(1-\lambda\overline{X})} + \frac{2\overline{X}}{(1+\lambda\overline{X})}$$

2. Consider a M/G/1 queue with three priority classes where class 3 has the highest priority and class 1 the lowest. For the *i*th class, the job arrival rate is λ_i with $\overline{X_i^n}$ as the *n*th moment of its service time. It is given that class 3 and class 2 have pre-emptive resume priority over class 1 but class 3 has only non-preemptive priority over class 2. Use the *Residual Life* approach to find the mean time spent in system by a job for each class.

We consider each class separately, starting from the highest priority class

Class 3: Mean Residual Lifetime $R_3 = \frac{1}{2}\left(\lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)$

$$W_{q3} = R_3 + \overline{X}_3 N_{q3} \quad \Rightarrow \quad W_{q3} = \frac{R_3}{(1-\rho_3)} \quad \text{with } \rho_3 = \lambda_3 \overline{X}_3$$

$$\boxed{W_3 = \overline{X}_3 + \frac{1}{2(1-\rho_3)}\left(\lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)}$$

Class 2: Mean Residual Lifetime $R_2 = R_3 = \frac{1}{2}\left(\lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)$

$$W_{q2} = R_2 + N_{q3}\overline{X}_3 + \overline{X}_3 \lambda_3 W_{q2} + \overline{X}_2 N_{q2}$$

$$\Rightarrow \quad W_{q2} = \frac{R_2 + \rho_3 W_{q3}}{(1-\rho_2-\rho_3)} \quad \text{with } \rho_2 = \lambda_2 \overline{X}_2$$

$$\boxed{W_2 = \overline{X}_2 + \frac{1}{(1-\rho_2-\rho_3)}\left[\frac{\left(\lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)}{2} + \rho_3 W_{q3}\right]}$$

Class 1: Mean Residual Lifetime $R_1 = \frac{1}{2}\left(\lambda_1 \overline{X_1^2} + \lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)$

$$W_1 = \overline{X}_1 + \frac{R_1}{(1-\rho_1-\rho_2-\rho_3)} + \overline{X}_3 \lambda_3 W_1 + \overline{X}_2 \lambda_2 W_1 \quad \text{with } \rho_1 = \lambda_1 \overline{X}_1$$

$$\Rightarrow \boxed{W_1 = \frac{\overline{X}_1(1-\rho_1-\rho_2-\rho_3) + R_1}{(1-\rho_2-\rho_3)(1-\rho_1-\rho_2-\rho_3)}}$$

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As for the usual M/G/1 queue, we can write $W_q = \frac{R}{(1-\lambda\overline{X})}$. Consider an interval (0, *t*) where $t \to \infty$ and let *M*(*t*)= No. of arrivals in (0, *t*) and *N*(*t*)= No. of busy periods in (0, *t*).

Average Length of a Busy Period = $\frac{\overline{X}}{1-\lambda\overline{X}}$

Mean Cycle Time = $T_C = \frac{1}{\lambda} + \frac{\overline{X}}{1-\lambda\overline{X}} = \frac{1}{\lambda(1-\lambda\overline{X})}$

Therefore $N(t) = \frac{t}{T_C} = t\lambda(1-\lambda\overline{X})$

Note that $R_t = \frac{1}{t}\int_0^t r(\tau)d\tau = \frac{1}{2}\left[\frac{1}{t}\sum_{i=1}^{M(t)-N(t)} X_i^2 + \frac{1}{t}\sum_{j=1}^{N(t)} X_j^{*2}\right]$, where X_j^* is the length of the first service time of the *j*[th] busy period.

Therefore $R_t = \frac{1}{2}\left[\frac{(M-N)}{t}\left(\frac{1}{M-N}\right)\sum_{i=1}^{M(t)-N(t)} X_i^2 + \frac{N}{t}\left(\frac{1}{N}\right)\sum_{j=1}^{N(t)} X_j^{*2}\right]$

Taking limits as $t \to \infty$, we get -

$\lim_{t\to\infty}\frac{N(t)}{t} = \lambda(1-\lambda\overline{X})$ and $\lim_{t\to\infty}\frac{M(t)-N(t)}{t} = \lambda - \lambda(1-\lambda\overline{X}) = \lambda^2\overline{X}$

and

$$R = \frac{1}{2}\left[\lambda^2\overline{X}\left(\overline{X^2}\right) + \lambda(1-\lambda\overline{X})\left(2\overline{X^2}\right)\right]$$

$$= \frac{\lambda\overline{X^2}}{2}\left[2-\lambda\overline{X}\right]$$

Therefore,

$$\boxed{W_q = \frac{\lambda\overline{X^2}}{2}\left(\frac{2-\lambda\overline{X}}{1-\lambda\overline{X}}\right)}$$

Mean Service Time = $\overline{X}$

Therefore,

$$\boxed{W = \frac{\lambda\overline{X^2}}{2}\left(\frac{2-\lambda\overline{X}}{1-\lambda\overline{X}}\right) + \overline{X}}$$

2. Consider a M/G/1 queue with three priority classes where class 3 has the highest priority and class 1 the lowest. For the i^{th} class, the job arrival rate is λ_i with $\overline{X_i^n}$ as the n^{th} moment of its service time. It is given that class 3 has pre-emptive resume priority over class 2 and class 1 but class 2 has only non-preemptive priority over class 1. Use the *Residual Life* approach to find the mean time spent in system by a job for each class.

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$$\boxed{W_3 = \overline{X}_3 + \frac{1}{2(1-\rho_3)}\left(\lambda_3 \overline{X_3^2}\right)}$$

Class 2: Mean Residual Lifetime $R_2 = \frac{1}{2}\left(\lambda_1 \overline{X_1^2} + \lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)$

$$W_2 = \overline{X}_2 + \frac{R_2}{(1-\rho_2-\rho_3)} + \overline{X}_3 \lambda_3 W_2$$

with $\rho_2 = \lambda_2 \overline{X}_2$

$$\Rightarrow \boxed{W_2 = \frac{\overline{X}_2(1-\rho_2-\rho_3) + R_2}{(1-\rho_3)(1-\rho_2-\rho_3)}}$$

Class 1: Mean Residual Lifetime $R_1 = R_2 = \frac{1}{2}\left(\lambda_1 \overline{X_1^2} + \lambda_2 \overline{X_2^2} + \lambda_3 \overline{X_3^2}\right)$

$$W_1 = \overline{X}_1 + \frac{R_1}{(1-\rho_1-\rho_2-\rho_3)} + \overline{X}_3 \lambda_3 W_1 + \overline{X}_2 \lambda_2 W_1 \quad \text{with } \rho_1 = \lambda_1 \overline{X}_1$$

$$\Rightarrow \boxed{W_1 = \frac{\overline{X}_1(1-\rho_1-\rho_2-\rho_3) + R_1}{(1-\rho_2-\rho_3)(1-\rho_1-\rho_2-\rho_3)}}$$