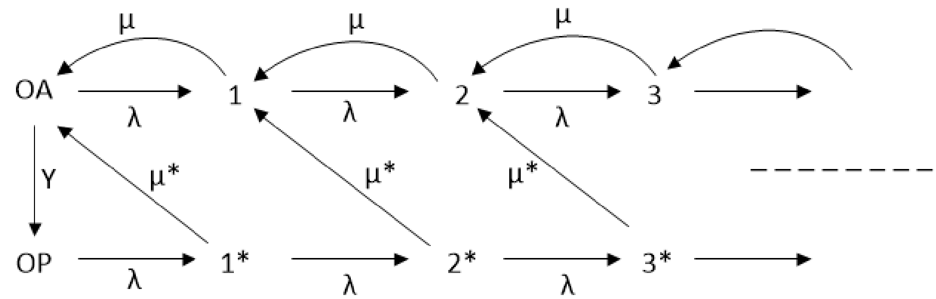
EE 633, Queueing Systems (2016-17F)
Solution to the Mid-Term Examination

1.

(a) State Transition Diagram



(b) Balance Equations

$$\gamma p_{OA} = \lambda p_{OP}$$
$$(\lambda+\mu^*) p_{1*} = \lambda p_{OP}$$
$$\mu p_1 = \lambda\left(p_{OA}+p_{1*}\right) = \lambda\left[\frac{\lambda}{\gamma}+\frac{\lambda}{\lambda+\mu^*}\right] p_{OP}$$
$$(\lambda+\mu^*) p_{n*} = \lambda p_{(n-1)*} \quad n\ge 2$$
$$\lambda(p_{OA}+p_{1*}) = \mu p_1$$
$$\lambda(p_1+p_{2*}) = \mu p_2$$
....
$$\lambda(p_{n-1}+p_{n*}) = \mu p_n \quad n\ge 2$$

$$p_{OA} = \frac{\lambda}{\gamma} p_{OP}$$
$$p_{1*} = \frac{\lambda}{\lambda+\mu^*} p_{OP} \qquad p_1 = \frac{\lambda^2(\lambda+\mu^*+\gamma)}{\mu\gamma(\lambda+\mu^*)} p_{OP}$$
$$p_{n*} = \left(\frac{\lambda}{\lambda+\mu^*}\right)^n p_{OP} \quad n\ge 1$$

(c) Generating Functions

Since $p_{n*} = \left(\frac{\lambda}{\lambda+\mu^*}\right)^n p_{OP} \quad n\ge 1$, we get that $P^*(z) = \frac{\lambda z}{\lambda(1-z)+\mu^*} p_{OP}$

and $\lambda p_{OA} z + \lambda P^*(z) + \lambda z P(z) = \mu P(z)$

$$(\mu-\lambda z)P(z) - \lambda P^*(z) = \lambda p_{OA} z = \frac{\lambda^2 z}{\gamma} p_{OP}$$

$$\frac{(\mu-\lambda z)}{\lambda^2 z} P(z) = \left[\frac{\lambda(1-z)+\mu^*+\gamma}{\left(\lambda(1-z)+\mu^*\right)\gamma}\right] p_{OP}$$

$$P(z) = \frac{\lambda^2 z}{(\mu-\lambda z)}\left[\frac{\lambda(1-z)+\mu^*+\gamma}{\left(\lambda(1-z)+\mu^*\right)\gamma}\right] p_{OP}$$
$$= \frac{\lambda^2 z}{(\mu-\lambda z)}\left[\frac{1}{\gamma}+\frac{1}{\left(\lambda(1-z)+\mu^*\right)}\right] p_{OP}$$

(d) Note that $P(1)$=P{server working with rate μ} and $P^*(1)$=P{server working with rate μ^*}. Once we find these, we can find –

Fraction of the Busy Period when server is working with rate $\mu^* = \frac{P^*(1)}{P(1)+P^*(1)}$

From the results of (c), we get

$$P^*(1) = \frac{\lambda}{\mu^*} p_{OP} \quad P(1) = \frac{\lambda^2}{(\mu-\lambda)}\left[\frac{1}{\gamma} + \frac{1}{\mu^*}\right] p_{OP} = \frac{\lambda^2\left(\gamma+\mu^*\right)}{\gamma\mu^*(\mu-\lambda)} p_{OP}$$

$$P(1) + P^*(1) = \frac{\lambda(\lambda\mu^* + \mu\gamma)}{(\mu-\lambda)\gamma\mu^*} p_{OP}$$

Therefore, $$\frac{P^*(1)}{P(1)+P^*(1)} = \frac{(\mu-\lambda)\gamma}{\gamma\mu + \lambda\mu^*}$$

(d) We can derive the following using the actual probability distributions, but it should be evident that when $\lambda=\gamma$, 50% of the time the Busy Period will start with normal service and 50% of the time it will start with exceptional first service.

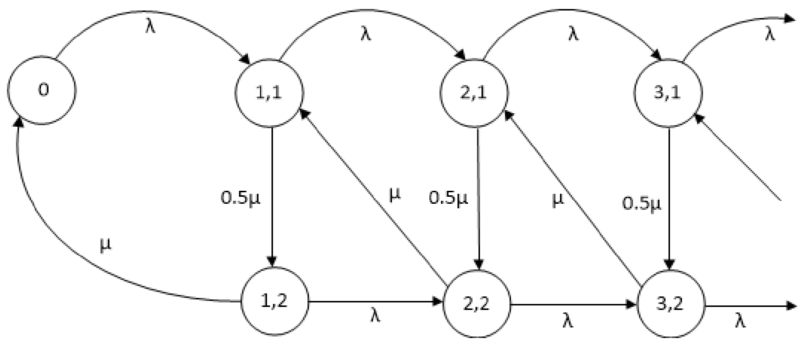
Therefore, the mean length of the Busy Period will be –

$$\overline{BP} = \frac{1}{2}\left[\frac{\frac{1}{\mu}}{1-\frac{\lambda}{\mu}} + \frac{1}{\mu^*} + \frac{\lambda}{\mu^*}\frac{\frac{1}{\mu}}{1-\frac{\lambda}{\mu}}\right] = \frac{1}{2}\frac{(\mu^*+\mu)}{\mu^*(\mu-\lambda)}$$

Therefore $$p_0 = \frac{\bar{I}}{\bar{I} + \overline{BP}} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{1}{2}\frac{(\mu^*+\mu)}{\mu^*(\mu-\lambda)}} = \frac{2\left(1-\frac{\lambda}{\mu}\right)}{2+\lambda\left(\frac{1}{\mu^*} - \frac{1}{\mu}\right)}$$

2.

(a) State Transition Diagram



(b) Balance Equations

$$\mu p_{12} = \lambda p_0$$
$$(\lambda+\mu) p_{12} = 0.5\mu p_{11}$$
$$(\lambda+\mu) p_{22} = 0.5\mu p_{21} + \lambda p_{12} \qquad\qquad \lambda(p_{n1}+p_{n2}) = \mu p_{n+1,2} \quad n \ge 1$$
$$\cdots\cdots\cdots$$
$$(\lambda+\mu) p_{n2} = 0.5\mu p_{n1} + \lambda p_{n-1,2} \quad n \ge 2$$

(c) Note that $P(z) = \sum_{n=0}^{\infty} p_n z^n = p_0 + P_1(z) + P_2(z)$ where $P_1(z) = \sum_{n=1}^{\infty} p_{n,1} z^n$ and $P_2(z) = \sum_{n=1}^{\infty} p_{n,2} z^n$

$$(\lambda+\mu) P_2(z) = 0.5\mu P_1(z) + \lambda z P_2(z) \qquad\qquad \lambda[P_1(z)+P_2(z)] = \frac{\mu}{z}[P_2(z) - p_{12} z]$$
$$P_2(z) = \frac{0.5}{1+\rho-\rho z} P_1(z) \qquad\qquad (1-\rho z) P_2(z) - \rho z P_1(z) = \rho z p_0$$

(d) The Normalization Condition would be $p_0 + P_2(1) + P_1(1) = 1$

We have $\quad P_2(1) = 0.5 P_1(1) \qquad (1-\rho)P_2(1) - \rho P_1(1) = \rho p_0$

Solving, we get $\quad P_1(1) = \frac{2\rho}{1-3\rho} p_0 \qquad P_2(1) = \frac{\rho}{1-3\rho} p_0$

Therefore $\quad p_0\left(1 + \frac{2\rho}{1-3\rho} + \frac{\rho}{1-3\rho}\right) = 1 \quad \Rightarrow \quad p_0 = 1-3\rho$

(e) The easiest way to do this is by using the Residual Life result $W_q = \frac{\lambda \overline{X^2}}{2(1-\lambda \bar{X})}$ and then use Little's Result $N_q = \lambda W_q$ to find N_q. To find $\bar{X}$ and $\overline{X^2}$, we need to find $L_B(s)$, the Laplace Transform of the overall service time distribution.

$$L_B(s) = \frac{0.5\mu^2}{(s+\mu)^2} + \frac{0.5\mu}{(s+\mu)} L_B(s) \quad \Rightarrow \quad L_B(s) = \frac{\mu^2}{2s^2 + 3\mu s + \mu^2}$$

Then

$$L_B'(s) = -\frac{\mu^2(4s+3\mu)}{(2s^2+3\mu s+\mu^2)^2} \qquad \bar{X} = -L_B'(0) = \frac{3}{\mu}$$

$$L_B''(s) = -\frac{4\mu^2}{(2s^2+3\mu s+\mu^2)^2} + \frac{2\mu^2(4s+3\mu)^2}{(2s^2+3\mu s+\mu^2)^3} \qquad \overline{X^2} = L_B''(0) = \frac{14}{\mu^2}$$

Therefore, $$W_q = \frac{1}{2}\lambda \frac{\left(\frac{14}{\mu^2}\right)}{\left(1-\frac{3\lambda}{\mu}\right)} = \frac{7\lambda}{\mu(\mu-3\lambda)} = \frac{7\rho^2}{\lambda(1-3\rho)} \qquad \Rightarrow \qquad N_q = \frac{7\rho^2}{1-3\rho}$$