EE 679, Queueing Systems (2001-02F)
Solutions to Test -4

1. Average Busy Period length without exceptional first service = $\dfrac{\overline{X}}{1-\lambda\overline{X}}$

Average Busy Period Length for this queue $= \overline{X}+\overline{\Delta}+\lambda(\overline{X}+\overline{\Delta})\dfrac{\overline{X}}{1-\lambda\overline{X}}$

$$= \frac{\overline{X}+\overline{\Delta}}{1-\lambda\overline{X}}$$

Average Idle Period Length = $\dfrac{1}{\lambda}$

Average Cycle Length = $\dfrac{1}{\lambda}+\dfrac{\overline{X}+\overline{\Delta}}{1-\lambda\overline{X}}=\dfrac{1+\lambda\overline{\Delta}}{\lambda(1-\lambda\overline{X})}$

P{system is empty} = $p_0=\dfrac{Average\ \ Idle\ Period\ Length}{Average\ \ Cycle\ Length}=\dfrac{1-\lambda\overline{X}}{1+\lambda\overline{\Delta}}$

We use the residual life approach here to find the mean queueing delay W_q. Proceeding as in Chapter 4, consider the time interval *(0, t)* where we would eventually let $t\to\infty$. Let M be the number of service completions in this interval and N the number of cycles. (We can ignore end effects and incomplete intervals, as we will only consider the case $t\to\infty$.) We can then write the following.

$$R_t=\frac{1}{t}\int_0^t r(t)dt=\frac{1}{2}\left[\frac{(M-N)}{t}\frac{1}{(M-N)}\sum_{i=1}^{M-N}X_i^2+\frac{N}{t}\frac{1}{N}\sum_{j=1}^{N}(X_j+\Delta)^2\right]$$

Taking limits as $t\to\infty$, we can then show that the mean residual service time R will be

$$R=\frac{1}{2}\lambda\overline{X^2}+\frac{1}{2}\lambda\frac{(\overline{\Delta^2}+2\overline{X}\,\overline{\Delta})(1-\lambda\overline{X})}{1+\lambda\overline{\Delta}}$$

Note that to get the above, we use our earlier results to show that

$$\lim_{t\to\infty}\frac{N}{t}=\frac{\lambda(1-\lambda\overline{X})}{1+\lambda\overline{\Delta}} \qquad \text{and} \qquad \lim_{t\to\infty}\frac{M}{t}=\lambda$$

The mean queueing delay W_q may be obtained as in Chapter 4 to be

$$W_q=\frac{R}{1-\lambda\overline{X}}=\frac{1}{2}\frac{\lambda\overline{X^2}}{1-\lambda\overline{X}}+\frac{1}{2}\frac{\lambda(\overline{\Delta^2}+2\overline{X}\,\overline{\Delta})}{1+\lambda\overline{\Delta}}$$

It should be noted that the mean service time $\overline{X^*}$ in this case is given by

$$\overline{X^*} = p_0(\overline{X} + \overline{\Delta}) + (1 - p_0)\overline{X} = \frac{\overline{X} + \overline{\Delta}}{1 + l\overline{\Delta}}$$

Therefore, we can find the mean time W spent in the system by an arriving customer as

$$W = W_q + \overline{X^*}$$

using the values of W_q and $\overline{X^*}$ obtained earlier.

2. For this system, the imbedded Markov Chain at the customer departure instants may be written as

$$\begin{aligned} n_{i+1} &= a^*_{i+1} && \text{for } n_i = 0,\ 1 \\ &= n_i + a_{i+1} - 1 && \text{for } n_i \text{ ³ } 2 \end{aligned}$$

Taking expectations of both sides of the above at equilibrium and using $r = l\overline{X}$ and $r^* = l\,\overline{X^*}$, we get

$$N = (p_0 + p_1)r^* + (N - p_1) + (1 - p_0 - p_1)(r - 1)$$

which may be simplified to get

$$p_0(r^* - r + 1) + p_1(r^* - r) = 1 - r \tag{A}$$

The generating function $P(z)$ for the number in the system may be found in the usual way from the imbedded Markov Chain under equilibrium conditions. This may be found to be

$$P(z) = \frac{zA^*(z)(p_0 + p_1) - A(z)(p_0 + p_1 z)}{z - A(z)}$$
$$\textit{with} \quad A(z) = \tilde{B}(l - lz),\ A^*(z) = \tilde{B}^*(l - lz)$$

Using

$$a_0 = A(0) = \tilde{B}(l) \quad a_1 = A'(0) = -l\tilde{B}'(l) \quad a^*_0 = A^*(0) = \tilde{B}^*(l) \quad a^*_1 = A^{*\prime}(0) = -l\tilde{B}^{*\prime}(l)$$

and the fact that $P'(0)=p_1$, we get that

$$P'(0) = p_1 = -\frac{p_0 a_0(a_1 - 1)}{a_0^2} + \frac{a^*_0(p_0 + p_1) - a_1 p_0 - a_0 p_1}{(-a_0)}$$

This may be simplified to get

$$p_1 = p_0 \frac{1 - a^*_0}{a^*_0} \tag{B}$$

Solving (A) and (B), we get

$$p_0 = \frac{(1-r)a_0^*}{r^* - r + a_0^*}$$

$$p_1 = \frac{(1-r)(1-a_0^*)}{r^* - r + a_0^*}$$

as required to be found.

Note that with the parameters as given above, we can now also find the actual expression for *P(z)* and use that generating function to find the moments of the number in the system. For a FCFS queue, this may also be used in the standard way to find the distribution of the time spent in system. The moment of this delay parameter may then be found using this or may be found by applying Little's result to the mean number in the system.