EE 679, Queueing Systems (2000-01F)
Solutions to Test -1

1. P{system fails in the interval (t, t+dt)}= $f_X(t)dt = P\{X > t\}b(t)dt$

Therefore, $f_X(t)dt = [1 - F_X(t)]b(t)dt$

Hence, $$\frac{dF_X(t)}{dt} = [1 - F_X(t)]b(t) \qquad \textbf{(1A)}$$

Note that the initial condition for the above will have to be $F_X(0) = 0$ since the system starts operation at t=0

(a) Given *b(t)=kt* we get from **(1A)** that, $[1 - F_X(t)] = Ke^{-\frac{1}{2}kt^2}$ for *t³0*

Using the initial condition, $F_X(0) = 0$, we can then obtain K=1

Therefore $F_X(t) = 1 - e^{-\frac{1}{2}kt^2}$ for *t³0*

Hence $F_X(x) = 1 - e^{-\frac{1}{2}kx^2}$ or $\boxed{f_X(x) = kxe^{-\frac{1}{2}kx^2} \text{ for } x³0}$ *[5]*

(b) Since X is uniformly distributed over (0,T), therefore -

$$f_X(t) = \frac{1}{T} \qquad \text{for } 0 \le t \le T$$

Therefore $$F_X(t) = \frac{t}{T} \qquad \text{for } 0 \le t \le T$$

Substituting in **(1A)**, we get -

$$\frac{1}{T} = (1 - \frac{t}{T})b(t) \Rightarrow \boxed{b(t) = \frac{1}{(T-t)} \text{ in } 0 \le t \le T}$$ *[5]*

2. **(a)** E{z^N | $Y=y$, $N=n$ arrivals in y}= z^n

But P{n arrivals in y}=$\frac{(ly)^n}{n!}e^{-ly}$ Poisson Distribution

Therefore, $$E\{z^N \mid Y=y\} = \sum_{n=0}^{\infty} z^n \frac{(ly)^n}{n!}e^{-ly} = e^{-ly(1-z)}$$

Hence $$G_N(z) = E\{z^N\} = \int_{y=0}^{\infty} f_Y(y)E\{z^N \mid Y=y\}dy = \int_{y=0}^{\infty} f_Y(y)e^{-ly(1-z)}dy$$

Since $\tilde{F}_Y(s) = \int_{y=0}^{\infty} f_Y(y)e^{-sy}dy$, therefore $\boxed{G_N(z) = \tilde{F}_Y(l - lz)}$ *[5]*

(b) Using $s = l - l z$ wherever needed below -

$$\overline{N} = \left.\frac{dG_N(z)}{dz}\right|_{z=1} = -l \left.\frac{d\tilde{F}_Y(s)}{ds}\right|_{s=0} = l\overline{Y} \qquad \boxed{\overline{N} = l\overline{Y}} \qquad [2]$$

$$\overline{N(N-1)} = \left.\frac{d^2 G_N(z)}{dz^2}\right|_{z=1} = l^2\overline{Y^2} \qquad \text{or} \qquad \boxed{\overline{N^2} = l^2\overline{Y^2} + l\overline{Y}} \qquad [3]$$

(c) Given that $f_Y(y) = me^{-my}$ for $y^{3}0$

We can find that $\tilde{F}_Y(s) = \dfrac{m}{s+m}$

Therefore

$$G_N(z) = \tilde{F}_Y(l - lz) = \frac{m}{(l+m) - lz} = \left(\frac{m}{l+m}\right)\frac{1}{\left[1 - \dfrac{l}{l+m}z\right]}$$

or $$G_N(z) = \left(\frac{m}{l+m}\right)\sum_{n=0}^{\infty}\left(\frac{l}{l+m}z\right)^n = \sum_{n=0}^{\infty}\left[\frac{ml^n}{(l+m)^{n+1}}\right]z^n$$

Therefore $$\boxed{P\{N = n\} = \frac{ml^n}{(l+m)^{n+1}}} \qquad [5]$$

Note: Only 2 marks will be given if answer left in form of $G_N(z)$