EE633 2014-2015F
Quiz – I

Time: 45 minutes *Marks 10*

1. Circular on Usage of VIP Toilets (as issued by the *Competent Toilet Authority*)

Henceforth, the VIP toilet in the office will be made available for non-VIPs subject to the following rules.

i. No non-VIP user will be allowed to wait if the toilet is currently occupied by a VIP or non-VIP user
ii. If a VIP user arrives while a non-VIP user is using the toilet, the non-VIP user will be **immediately** evicted and the VIP user will be allowed to use the toilet
iii. VIP users can wait (infinite waiting space available) if another VIP user is currently using the toilet

You have been deputed by the Office Audit Team to study the effectiveness of these rules when the system is operating at equilibrium. You have determined that VIP users arrive from a Poisson process with average arrival rate λ_V and occupy the toilet for an exponentially distributed time with mean $1/\mu_V$. Non-VIP users arrive from a Poisson process with average arrival rate λ_N and occupy the toilet for an exponentially distributed time with mean $1/\mu_N$. (For Notational Convenience, $\rho_V = \frac{\lambda_V}{\mu_V},\ \rho_N = \frac{\lambda_N}{\mu_N}$)

Denote the system state by {Number of VIP users in the system, Number of Non-VIP users in the system}

a) What is the probability that a VIP user will have to wait to use the toilet? **[1]**
b) Draw the State Transition Diagram for the system **[2]**
c) Write the Balance Equations and solve them to obtain the state probabilities **[4]**
d) What is the overall efficiency of toilet usage? (i.e. fraction of time toilet is in use) **[0.5]**
e) What is ratio of toilet usage of non-VIP customers to that of VIP customers **[0.5]**

2. Due to the criticism levelled on the scheme of **(1)**, the *Competent Toilet Authority* has asked you for advice on modifying the scheme to make it fairer for the non-VIP users while retaining some of the VIP privileges. You have advised that the following changes be made to the scheme of (1) –

i. Non-VIP users are now allowed to wait for the toilet if there are no VIP users in the system.
ii. If a non-VIP user is using the toilet when a VIP customer arrives, the non-VIP user will be allowed to complete his/her toilet usage (i.e. will not be forced to leave immediately!). However, all other non-VIP users will be forced to leave immediately. New non-VIP users will also not be allowed into the toilet system until all VIP users have left.

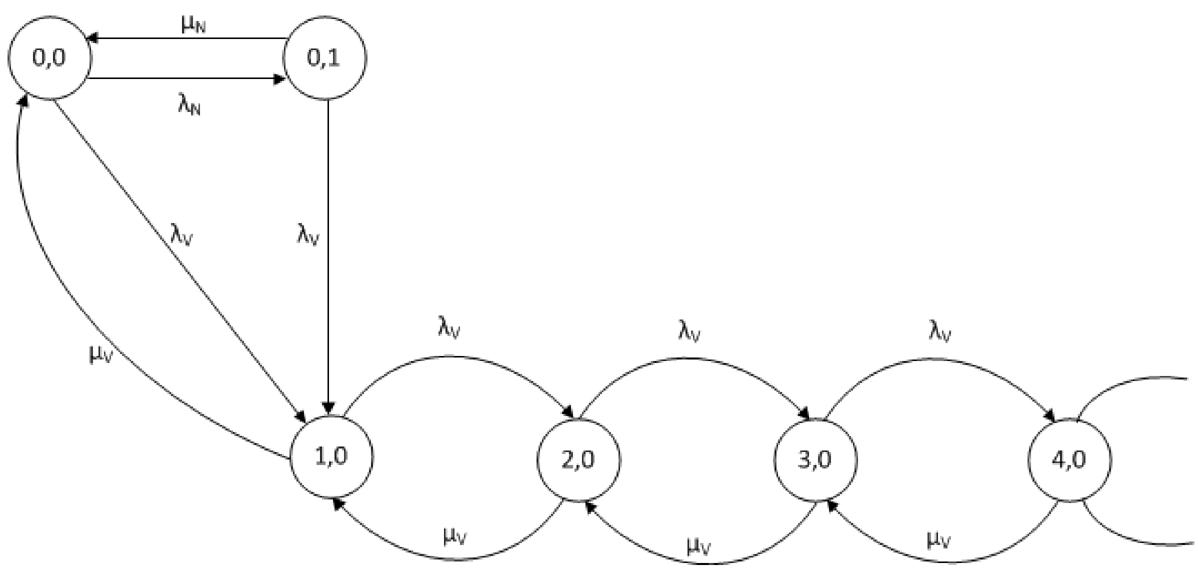
Draw the State Transition Diagram for this modified system **[2]**

Solutions:

(a) Note that the VIP users are not affected at all by the non-VIP users. Therefore, for the VIP users, this is just a simple M/M/1 queue with offered traffic of ρ_V. Therefore the probability that the system has no VIP customers is (1- ρ_V).

Therefore P{VIP customer has to wait to use the toilet}= ρ_V

(b) State Transition Diagram



(c) Balance Equations and State Probabilities

$$p_{0,1}(\lambda_V + \mu_N) = p_{0,0}\lambda_N \qquad \Rightarrow p_{0,0} = \frac{\lambda_V + \mu_N}{\lambda_N} p_{0,1}$$

$$\lambda_V (p_{0,0} + p_{0,1}) = \mu_V p_{1,0} \qquad \Rightarrow p_{0,0} + p_{0.1} = \frac{p_{1,0}}{\rho_V}$$

$$\lambda_V p_{n,0} = \mu_V p_{n+1,0} \quad n = 1, 2, 3, ...$$

It is evident that $p_{n,0} = \rho_V^{n-1} p_{1,0} \quad n = 1, 2, 3, ...$

Therefore $p_{1,0} + p_{2,0} + p_{3,0} + ... = \sum_{n=1}^{\infty} p_{n,o} = \frac{p_{1,0}}{1-\rho_V}$

Since $p_{0,0} + p_{0,1} = 1 - (p_{1,0} + p_{2,0} + p_{3,0} + ...) = 1 - \frac{p_{1,0}}{1-\rho_V}$ (*Normalization Condition)*

and $1 - \frac{p_{1,0}}{1-\rho_V} = \frac{p_{1,0}}{\rho_V}$

Therefore $p_{1,0} = \rho_V(1-\rho_V)$ (*Expected from first principles*)

Using the expression for $p_{1,0}$ we get $p_{0,0} + p_{0,1} = \frac{p_{1,0}}{\rho_V} = 1-\rho_V$ & $p_{0,0} = \frac{\lambda_V + \mu_N}{\lambda_N} p_{0,1}$

Therefore, $p_{0,1}\left(\frac{\lambda_N + \mu_N + \lambda_V}{\lambda_N}\right) = (1-\rho_V)$

$$p_{0,1} = \frac{(1-\rho_V)\lambda_N}{\lambda_N + \mu_N + \lambda_V} \quad p_{0,0} = \frac{(1-\rho_V)(\lambda_V + \mu_N)}{\lambda_N + \mu_N + \lambda_V}$$

and $p_{n,0} = \rho_V^n(1-\rho_V) \quad n = 1,2,3,.........$

(d) Efficiency of toilet usage = P{toilet is not empty} =

$$1 - p_{0,0} = 1 - \frac{(1-\rho_V)(\lambda_V + \mu_N)}{\lambda_N + \mu_N + \lambda_V} = \frac{(\lambda_V + \mu_N)\rho_V + \lambda_N}{\lambda_N + \mu_N + \lambda_V}$$

(e) Ratio of toilet usage of non-VIP customers to that of VIP customers =

$$\frac{p_{0,1}}{\sum_{n=1}^{\infty} p_{n,0}} = \frac{\frac{(1-\rho_V)\lambda_N}{\lambda_N + \mu_N + \lambda_V}}{\rho_V} = \left(\frac{1-\rho_V}{\rho_V}\right)\left(\frac{\lambda_N}{\lambda_N + \mu_N + \lambda_V}\right)$$

2. State Transition Diagram of the Modified System (same definition of the system state)