EE 633
Solutions to Quiz-I

(a) *T* is fixed

Mean Length of Server Idle in a cycle = $\overline{I} = T + e^{-\lambda T}\frac{1}{\lambda} = \frac{e^{-\lambda T} + \lambda T}{\lambda}$ **[1]**

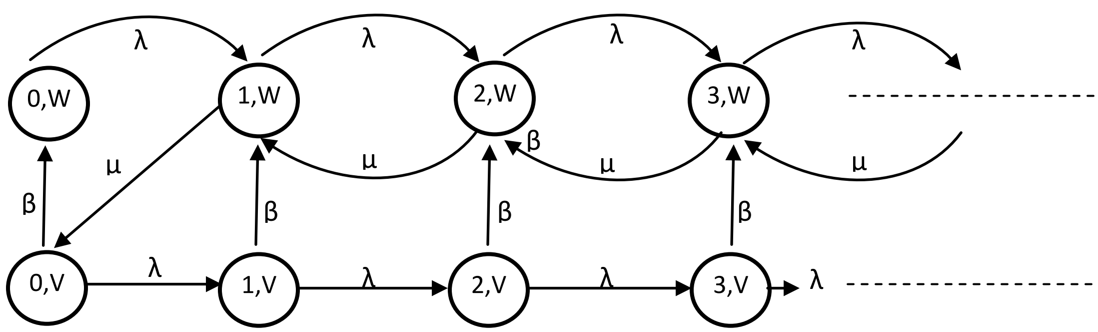
Mean Length of Busy Period in a cycle

= $\overline{BP} = \sum_{n=1}^{\infty}\left(\frac{n\overline{X}}{1-\rho}\right)\frac{(\lambda T)^n}{n!}e^{-\lambda T} + \left(\frac{\overline{X}}{1-\rho}\right)e^{-\lambda T} = \left(\frac{\overline{X}}{1-\rho}\right)\left(\lambda T + e^{-\lambda T}\right)$ **[1]**

P{server idle} = $\frac{\overline{I}}{\overline{I} + \overline{BP}} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{\overline{X}}{1-\rho}} = 1-\rho$ **[1]**

(b) *T* is exponentially distributed with mean β^{-1}

(i) The State Transition Diagram is given below. **[3]**



The states (1,W), (2,W) ……etc. are the normal states with the server working. The states (*k*,V) are ones where the server is on its single vacation and *k* arrivals are in the system. The state (0,W) is the state where the server has completed its vacation (with no arrivals) and is now waiting for new arrivals to come in order to start service.

***P*(*V*) =** $\sum_{n=0}^{\infty} P(n,V)$ **and *P*(*W*)=** $\sum_{n=1}^{\infty} P(n,W)$ **may be found by solving for the state probabilities but it is possible to get these directly.** The solution method by finding the state probabilities is given later.

Using the result of (a)

$$\bar{I} = \int_0^\infty \left(\frac{e^{-\lambda T} + \lambda T}{\lambda} \right) \beta e^{-\beta T} dT = \frac{\beta}{\lambda(\beta+\lambda)} + \frac{1}{\beta} = \frac{\lambda^2 + \lambda\beta + \beta^2}{\lambda\beta(\lambda+\beta)}$$

$$\overline{BP} = \left(\frac{\bar{X}}{1-\rho} \right) \int_0^\infty \left(\lambda T + e^{-\lambda T} \right) \beta e^{-\beta T} dT = \left(\frac{\bar{X}}{1-\rho} \right) \left(\frac{\lambda}{\beta} + \frac{\beta}{(\beta+\lambda)} \right)$$

$$= \left(\frac{\bar{X}}{1-\rho} \right) \left(\frac{\lambda^2 + \lambda\beta + \beta^2}{\beta(\lambda+\beta)} \right)$$

Therefore, $\bar{T}_{cycle} = \bar{I} + \overline{BP} = \left(\frac{\lambda^2 + \lambda\beta + \beta^2}{\lambda\beta(\lambda+\beta)} \right) \frac{1}{(1-\rho)}$

Using the above and that the mean length of a vacation is 1/β, we get

(ii) $$P(V) = \frac{\bar{V}}{\bar{T}_{cycle}} = \frac{\frac{1}{\beta}}{\left(\frac{\lambda^2 + \lambda\beta + \beta^2}{\lambda\beta(\lambda+\beta)} \right) \frac{1}{(1-\rho)}} = \frac{\lambda(\lambda+\beta)(\mu-\lambda)}{\mu(\lambda^2 + \lambda\beta + \beta^2)}$$ **[3]**

(iii) $$P(W) = \frac{\overline{BP}}{\bar{T}_{cycle}} = \rho$$ **[1]**

Alternative Approach using Balance Equations

Balance Equations:

$$\beta p_{0V} = \lambda p_{0W} \qquad (\lambda+\beta) p_{0V} = \mu p_{1W} \qquad (\lambda+\mu) p_{1W} = \lambda p_{0W} + \mu p_{2W} + \beta p_{1V}$$
$$(\lambda+\beta) p_{1V} = \lambda p_{0V} \qquad (\lambda+\beta) p_{2V} = \lambda p_{1V} \;\ldots\ldots\; (\lambda+\beta) p_{(n+1)V} = \lambda p_{nV} \;\ldots\ldots$$
$$\lambda(p_{1W} + p_{1V}) = \mu p_{2W} \;\ldots\ldots\ldots\; \lambda(p_{nW} + p_{nV}) = \mu p_{(n+1)W} \;\ldots\ldots\ldots\ldots$$

Solving these

$$p_{0W} = \frac{\beta}{\lambda} p_{0V} \quad p_{1W} = \frac{\lambda+\beta}{\mu} p_{0V} \quad p_{1V} = \left(\frac{\lambda}{\lambda+\beta}\right) p_{0V} \;\ldots\ldots\; p_{nV} = \left(\frac{\lambda}{\lambda+\beta}\right)^{n} p_{0V} \;\ldots\ldots$$

Therefore, $P(V) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda+\beta}\right)^{n} p_{0V} = \frac{\lambda+\beta}{\beta} p_{0V}$ where $P(V)$ is the probability that the server is on vacation. To find $P(V)$ we still need to find p_{0V} which may be done using the normalization condition.

Note that $\lambda(p_{nW} + p_{nV}) = \mu p_{(n+1)W}$ for n=1,2,............. Summing these from n=1 to ∞ and denoting $P(W) = \sum_{n=1}^{\infty} p_{nW}$ for notational convenience, we get

$$\lambda P(W) + \lambda\left[\frac{\lambda+\beta}{\beta} - 1\right] p_{0V} = \mu P(W) - \mu \frac{\lambda+\beta}{\mu} p_{0V}$$

$$(\mu-\lambda) P(W) = \left(\frac{\lambda^2}{\beta} + \lambda + \beta\right) p_{0V} \qquad \Rightarrow P(W) = \frac{\lambda^2 + \lambda\beta + \beta^2}{(\mu-\lambda)\beta} p_{0V}$$

Normalization implies that $P(W) + P(V) + p_{0W} = 1$

Therefore,

$$p_{0V}\left(\frac{\lambda^2 + \lambda\beta + \beta^2}{(\mu-\lambda)\beta} + \frac{\lambda+\beta}{\beta} + \frac{\beta}{\lambda}\right) = 1$$

$$p_{0V} = \frac{\lambda\beta(\mu-\lambda)}{\mu\left(\lambda^2 + \lambda\beta + \beta^2\right)}$$

Using these, $P(V)$ and $P(W)$ may be found as

$$P(V) = \frac{\lambda(\lambda+\beta)(\mu-\lambda)}{\mu(\lambda^2 + \lambda\beta + \beta^2)} \qquad P(W) = \frac{\lambda^2 + \lambda\beta + \beta^2}{(\mu-\lambda)\beta} \frac{\lambda\beta(\mu-\lambda)}{\mu\left(\lambda^2 + \lambda\beta + \beta^2\right)} = \rho$$