EE 633, Queueing Systems (2016-17F)
Solutions to Quiz – II

Consider an M/G/1 queue where, if the idle period is longer than *T* (fixed), then **the first customer in the busy period following that idle period** requires special service with service time X^* (mean $\overline{X^*}$, second moment $\overline{X^{*2}}$, pdf $b^*(t)$ and LT of pdf $L_{B*}(s)$). All other customers are served with the normal service time X (mean $\overline{X}$, second moment $\overline{X^2}$, pdf $b(t)$ and LT of pdf $L_B(s)$). Consider the queue to be in equilibrium with arrivals coming from a Poisson process with average rate λ.

(a) Use the Busy Period approach to find -

(i) The probability of the server being idle

(ii) The overall mean service time X .

(i) $\overline{I} = \frac{1}{\lambda}$ $\qquad \overline{BP} = (1-e^{-\lambda T})\frac{\overline{X}}{1-\lambda\overline{X}} + e^{-\lambda T}\frac{\overline{X^*}}{1-\lambda\overline{X}} = \frac{\overline{X}+e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}{1-\lambda\overline{X}}$ $\qquad \overline{T}_{cycle} = \frac{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}{\lambda(1-\lambda\overline{X})}$

$$\Rightarrow \text{P\{server idle\}} = \frac{\overline{I}}{\overline{I}+\overline{BP}} = \frac{1-\lambda\overline{X}}{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}$$

(ii) Probability= $(1-e^{-\lambda T})$ Mean Length of BP= $\frac{\overline{X}}{1-\lambda\overline{X}}$ Mean Number= $\frac{1}{1-\lambda\overline{X}}$

Probability= $e^{-\lambda T}$ Mean Length of BP= $\frac{\overline{X^*}}{1-\lambda\overline{X}}$ Mean Number= $1+\frac{\lambda\overline{X^*}}{1-\lambda\overline{X}} = \frac{1+\lambda\left(\overline{X^*}-\overline{X}\right)}{1-\lambda\overline{X}}$

Considering both types of busy periods,

$\overline{BP} = \frac{\overline{X}+e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}{1-\lambda\overline{X}}$ Mean Number Served in Busy Period = $\frac{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}{1-\lambda\overline{X}}$

Therefore $\qquad X = \frac{\overline{X}+e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}$

(b) What will be the Mean Residual Service Time that will be observed by an arriving customer?

$$\overline{T}_{cycle} = \frac{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}{\lambda(1-\lambda\overline{X})}$$

Number of cycles in time interval of length *t* = $L(t) = \frac{t}{\overline{T}_{cycle}} = \frac{t\lambda(1-\lambda\overline{X})}{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}$

Of the total number of cycles, a fraction $e^{-\lambda T}$ will have a Busy Period where the first service is of length X^* and a fraction $(1-e^{-\lambda T})$ where the first service is of length X . Therefore, in a long interval of time *t*, if the total number of arrivals is $M(t)$, then there will be $N(t) = e^{-\lambda T} L(t)$ arrivals which will be served with service times, each of length X^*; the other M-N will be served with service time X. Using this,

$$N(t) = \frac{te^{-\lambda T}\lambda(1-\lambda\bar{X})}{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)} \qquad \Rightarrow \qquad \frac{N(t)}{t} = \frac{e^{-\lambda T}\lambda(1-\lambda\bar{X})}{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}$$

Therefore,

$$R = \lim_{t\to\infty}\left[\frac{1}{t}\sum_{i=1}^{M-N}\frac{X_i^2}{2} + \frac{1}{t}\sum_{j=1}^{N}\frac{X_j^{*2}}{2}\right] = \lim_{t\to\infty}\left[\frac{M-N}{t}\left(\frac{\overline{X^2}}{2}\right) + \frac{N}{t}\left(\frac{\overline{X^{*2}}}{2}\right)\right]$$

$$= \left(\lambda - \frac{e^{-\lambda T}\lambda(1-\lambda\bar{X})}{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}\right)\left(\frac{\overline{X^2}}{2}\right) + \left(\frac{e^{-\lambda T}\lambda(1-\lambda\bar{X})}{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}\right)\left(\frac{\overline{X^{*2}}}{2}\right)$$

$$= \frac{\lambda\overline{X^2}}{2} + \frac{e^{-\lambda T}\lambda(1-\lambda\bar{X})}{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}\left(\frac{\overline{X^{*2}}-\overline{X^2}}{2}\right)$$

(c) Write that State Transition Equations relating n_{i+1} to n_i for this queue.

$$\begin{aligned} n_{i+1} &= a_{i+1} & n_i &= 0,\ probability\ (1-e^{-\lambda T}) \\ &= a^*_{i+1} & n_i &= 0,\ probability\ e^{-\lambda T} \\ &= n_i + a_{i+1} - 1 & n_i &\geq 1 \end{aligned}$$

(d) Use the equation of (c) to confirm your result of part (i) of (a)

Taking expectations of both sides of the equation of (c) for a queue in equilibrium

$$\bar{n} = p_0\left[(1-e^{-\lambda T})\lambda\overline{X} + e^{-\lambda T}\lambda\overline{X^*}\right] + \bar{n} + (1-p_0)\left(\lambda\overline{X}-1\right)$$

$$p_0\left[1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)\right] = 1-\lambda\overline{X}$$

This gives $\quad p_0 = \dfrac{1-\lambda\bar{X}}{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}$ which is the same as the result obtained in part (i) of (a)

Bonus Questions

(e) $\quad W_q = N_q\overline{X} + R \qquad R = \dfrac{\lambda\overline{X^2}}{2} + \dfrac{e^{-\lambda T}\lambda(1-\lambda\bar{X})}{1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)}\left(\dfrac{\overline{X^{*2}}-\overline{X^2}}{2}\right)$

Therefore, $\quad W_q = \left(\dfrac{\lambda\overline{X^2}}{2(1-\lambda\overline{X})}\right) + \dfrac{\lambda e^{-\lambda T}\left(\overline{X^{*2}}-\overline{X^2}\right)}{2\left(1+\lambda e^{-\lambda T}\left(\overline{X^*}-\overline{X}\right)\right)}$

and $$W = W_q + X = \left(\frac{\lambda \overline{X^2}}{2(1-\lambda \overline{X})}\right) + \frac{\lambda e^{-\lambda T}\left(\overline{X^{*2}} - \overline{X^2}\right)}{2\left(1+\lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)\right)} + \frac{\overline{X} + e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)}{1+\lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)}$$

If you want to cross check with the answer of (f) – not required to be done!

$$N = \lambda W = \left(\frac{\lambda^2 \overline{X^2}}{2(1-\lambda \overline{X})}\right) + \frac{\lambda^2 e^{-\lambda T}\left(\overline{X^{*2}} - \overline{X^2}\right)}{2\left(1+\lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)\right)} + \frac{\lambda \overline{X} + \lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)}{1+\lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)}$$

(f) Using (c), we get -

$$P(z) = E\left\{z^n\right\} = p_0\left(1-e^{-\lambda T}\right)A(z) + p_0 e^{-\lambda T} A^*(z) + A(z)\sum_{n=1}^{\infty} z^{n-1} p_n$$

$$= p_0\left[A(z)\left(1-e^{-\lambda T}\right) + A^*(z)e^{-\lambda T} + \frac{A(z)}{z}\left\{P(z) - p_0\right\}\right]$$

Therefore,

$$P(z) = p_0\left[\frac{(z-1)A(z)}{z-A(z)} + e^{-\lambda T}\frac{z\left\{A^*(z) - A(z)\right\}}{z-A(z)}\right] \qquad A(z) = L_B(\lambda - \lambda z) \quad A^*(z) = L_{B^*}(\lambda - \lambda z)$$

Not required to be done

We can obtain p_0 from the above using $P(1) = 1$ or just use the one that we had obtained earlier. We can also use this to find the mean number in the system and cross-check the result we got in part (e).

$$(z-A)P = p_0\left[(z-1)A + e^{-\lambda T} z\left(A^* - A\right)\right]$$

$$(z-A)P' + (1-A')P = p_0\left[(z-1)A' + A + e^{-\lambda T}\left\{z(A^{*\prime} - A') + (A^* - A)\right\}\right]$$

Evaluating at *z*=1, we get

$$(1-\lambda \overline{X}) = p_0\left[1 + \lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)\right] \qquad \Rightarrow \qquad p_0 = \frac{1-\lambda \overline{X}}{1+\lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)}$$

$$(z-A)P'' + 2(1-A')P' - A''P = p_0\left[(z-1)A'' + 2A' + e^{-\lambda T}\left\{z(A^{*\prime\prime} - A'') + 2(A^{*\prime} - A')\right\}\right]$$

Evaluating at *z*=1, we get

$$2(1-\lambda \overline{X})N - \lambda^2 \overline{X^2} = p_0\left[2\lambda \overline{X} + \lambda^2 e^{-\lambda T}\left(\overline{X^{*2}} - \overline{X^2}\right) + 2\lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)\right]$$

$$= p_0\left[2\lambda\left\{\overline{X} + e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)\right\}\right] + p_0 \lambda^2 e^{-\lambda T}\left(\overline{X^{*2}} - \overline{X^2}\right)$$

Substituting for p_0 which was obtained earlier, we get

$$N = \lambda W = \left(\frac{\lambda^2 \overline{X^2}}{2(1-\lambda \overline{X})}\right) + \frac{\lambda^2 e^{-\lambda T}\left(\overline{X^{*2}} - \overline{X^2}\right)}{2\left(1+\lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)\right)} + \frac{\lambda \overline{X} + \lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)}{1+\lambda e^{-\lambda T}\left(\overline{X^*} - \overline{X}\right)}$$

Note that this matches what was obtained from the Residual Life Approach earlier