EE 633, Queueing Systems (2017-18F)
Solution to Quiz – I

(a) P{no arrival in vacation interval} = $\int_{t=0}^{\infty} e^{-\lambda t}\beta e^{-\beta t}dt = \dfrac{\beta}{\lambda+\beta}$

P{one or more arrivals in vacation interval} = $1-\dfrac{\beta}{\lambda+\beta}=\dfrac{\lambda}{\lambda+\beta}$

Therefore, **Mean Idle Time** $\bar{I}=\left(\dfrac{\beta}{\lambda+\beta}\right)\left(\dfrac{1}{\beta}+\dfrac{1}{\lambda}\right)+\left(\dfrac{\lambda}{\lambda+\beta}\right)\dfrac{1}{\beta}=\dfrac{(\lambda+\beta)^2-\lambda\beta}{\lambda\beta(\lambda+\beta)}$ **[5]**

(b) **Mean Busy Period** (see the steps given in the solution of Problem 4 in HA #2)

$$\overline{BP}=\left(\frac{\beta}{\lambda+\beta}\right)\left(\frac{1}{\mu-\lambda}\right)+\left(\frac{\lambda}{\lambda+\beta}\right)\left(\frac{\lambda+\beta}{\beta}\right)\left(\frac{1}{\mu-\lambda}\right)=\left(\frac{(\lambda+\beta)^2-\lambda\beta}{\beta(\lambda+\beta)}\right)\left(\frac{1}{\mu-\lambda}\right)$$ **[10]**

(c) **Mean Cycle Time** $\bar{T}_{cycle}=\left(\dfrac{(\lambda+\beta)^2-\lambda\beta}{\lambda\beta(\lambda+\beta)}\right)\left(1+\dfrac{\lambda}{\mu-\lambda}\right)=\left(\dfrac{(\lambda+\beta)^2-\lambda\beta}{\lambda\beta(\lambda+\beta)}\right)\left(\dfrac{1}{1-\rho}\right)$ **[5]**

Therefore, **Probability Server is Idle** = $\dfrac{\bar{I}}{\bar{T}_{cycle}}=(1-\rho)\quad \rho=\dfrac{\lambda}{\mu}$ (*expected, but steps must be shown)*

[5]

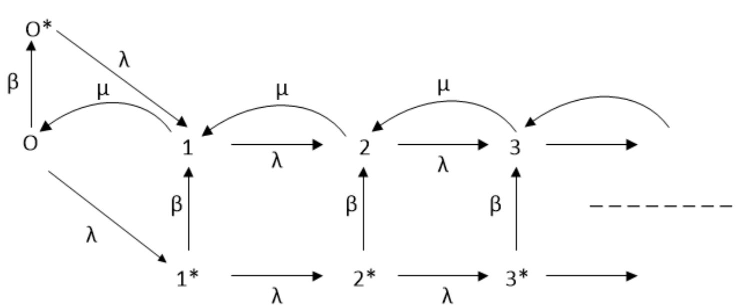
Probability Server is on Vacation = $\dfrac{1/\beta}{\bar{T}_{cycle}}=\left(\dfrac{\lambda(\lambda+\beta)}{(\lambda+\beta)^2-\lambda\beta}\right)(1-\rho)$ **[5]**

Verify these results later in (f) using the results of (e). Note that in terms of the results of (e),

P{server idle}= $p_0+p_{0*}+\sum_{n=1}^{\infty}p_{n*}$ *and* *P{server on vacation}=* $p_0+\sum_{n=1}^{\infty}p_{n*}$

(d) **State Transition Diagram**

[10]



(e) $p_0\beta=p_{0*}\lambda\quad p_{0*}=p_0\dfrac{\beta}{\lambda}\quad \left(p_0+p_{0*}\right)=\left(\dfrac{\lambda+\beta}{\lambda}\right)p_0$ **also** $P(z)=\sum_{n=1}^{\infty}z^np_n$ and $P^*(z)=\sum_{n=1}^{\infty}z^np_{n*}$

$$
\begin{array}{llll}
p_{1*}(\lambda+\beta)=\lambda p_0 & \lambda(p_0+p_{0*})=\mu p_1 & \lambda(p_1+p_{1*})=\mu p_2 & \times z \\
p_{2*}(\lambda+\beta)=\lambda p_{1*} & & \lambda(p_2+p_{2*})=\mu p_3 & \times z^2 \\
p_{3*}(\lambda+\beta)=\lambda p_{2*} & & \lambda(p_3+p_{3*})=\mu p_4 & \times z^3 \\
\ldots\ldots\vdots & & \ldots\ldots & \\
\ldots\ldots & & \ldots\ldots &
\end{array}
$$

$$
p_{n*}=\left(\frac{\lambda}{\lambda+\beta}\right)^n p_0 \qquad \vdots \qquad \lambda\left[P(z)+P^*(z)\right]=\frac{\mu}{z}\left[P(z)-p_1 z\right]
$$

[20+20]

$$
P^*(z)=p_0\frac{\lambda z}{\lambda+\beta-\lambda z} \qquad (\mu-\lambda z)P(z)-\lambda z P^*(z)=p_0(\lambda+\beta)
$$

$$
P^*(1)=p_0\frac{\lambda}{\beta} \qquad (\mu-\lambda)P(1)-\lambda P^*(1)=p_0(\lambda+\beta)
$$

$$
(\mu-\lambda)P(1)-p_0\frac{\lambda^2}{\beta}=p_0(\lambda+\beta)
$$

$$
P(1)=p_0\frac{\left[\lambda^2+\beta^2+\lambda\beta\right]}{\beta(\mu-\lambda)}=p_0\frac{\left[(\lambda+\beta)^2-\lambda\beta\right]}{\beta(\mu-\lambda)}
$$

Applying the Normalization Condition then gives,

$$
p_0\left[\frac{\lambda+\beta}{\lambda}+\frac{\lambda}{\beta}+\frac{(\lambda+\beta)^2-\lambda\beta}{(\mu-\lambda)\beta}\right]=1 \qquad \Rightarrow \qquad p_0=\frac{\lambda\beta(1-\rho)}{(\lambda+\beta)^2-\lambda\beta}
$$

[10]

This was not asked but you can use the above expressions to explicitly find the required generating functions. This gives us the following

$$
p_0=\frac{\lambda\beta(1-\rho)}{(\lambda+\beta)^2-\lambda\beta} \quad P^*(z)=p_0\frac{\lambda z}{\lambda+\beta-\lambda z} \quad P(z)=p_0\frac{z\left[(\lambda+\beta)^2-\lambda\beta z\right]}{(\mu-\lambda z)(\lambda+\beta-\lambda z)}
$$

(f) $p_0+p_{0*}=\dfrac{\beta(\lambda+\beta)(1-\rho)}{(\lambda+\beta)^2-\lambda\beta}$ and $\sum_{n=1}^{\infty}p_{n*}=P^*(1)=\dfrac{\lambda^2(1-\rho)}{(\lambda+\beta)^2-\lambda\beta}$

P{server not working} = $p_0+p_{0*}+\sum_{n=1}^{\infty}p_{n*}=\dfrac{(1-\rho)(\lambda\beta+\beta^2+\lambda^2)}{(\lambda+\beta)^2-\lambda\beta}=1-\rho$ **Confirmed** **[5]**

P{server on vacation} = $p_0+\sum_{n=1}^{\infty}p_{n*}=\dfrac{\lambda\beta(1-\rho)}{(\lambda+\beta)^2-\lambda\beta}+\dfrac{\lambda^2(1-\rho)}{(\lambda+\beta)^2-\lambda\beta}=\left(\dfrac{\lambda(\lambda+\beta)}{(\lambda+\beta)^2-\lambda\beta}\right)(1-\rho)$ **Confirmed**

[5]

A Note on the Solutions

Consider the situation where we choose the following sets of Balance Equations written for the states 1, 2, 3 .....

$$(\lambda+\mu)p_1 = \lambda p_{0*} + \beta p_{1*} + \mu p_2 \qquad \times z$$
$$(\lambda+\mu)p_2 = \lambda p_1 + \beta p_{2*} + \mu p_3 \qquad \times z^2$$
$$(\lambda+\mu)p_3 = \lambda p_2 + \beta p_{3*} + \mu p_4 \qquad \times z^3$$
....

....

Using these to generate generating functions, we get

$$(\lambda+\mu)P(z) = \lambda p_{0*} z + \lambda z P(z) + \beta P^*(z) + \frac{\mu}{z}\left[P(z) - p_1 z\right]$$

$$(\lambda+\mu)P(z) = \lambda p_{0*} z + \lambda z P(z) + \beta P^*(z) + \frac{\mu}{z}P(z) - \mu p_1$$

Note that $\mu p_1 = \lambda(p_0 + p_{0*}) = (\lambda+\beta)p_0$ and $p_{0*}\lambda = p_0\beta$. Therefore, we get

$$(\lambda+\mu)P(z) = \beta p_0 z + \lambda z P(z) + \beta P^*(z) + \frac{\mu}{z}P(z) - (\lambda+\beta)p_0$$

Evaluating this expression at $z=1$, we get once again $P^*(1) = p_0\dfrac{\lambda}{\beta}$

Therefore, this would not allow us to find a solution for $P(1)$ that we can use in the *Normalization Condition* to find p_0 . Therefore, this set of Balance Equations will not be of use to us.