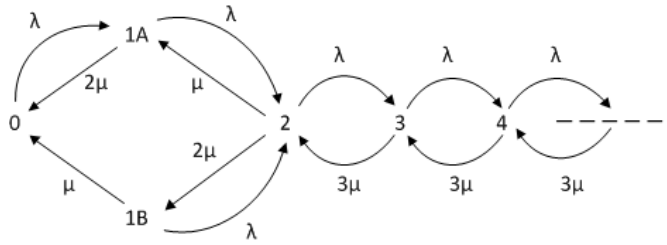
EE 633, Queueing Systems (2016-17F)
Solution to Quiz – I

(a) State Transition Diagram

States defined as 0: System Empty
1A: One user served by Teller A
1B: One user served by Teller B
n, n≥2 : Normal system state with both tellers working



(b) Balance Equations …… *one equation out of the first four is redundant!*

$$\lambda p_0 = 2\mu p_{1A} + \mu p_{1B}$$
$$(\lambda + 2\mu) p_{1A} = \lambda p_0 + \mu p_2$$
$$p_{1B}(\lambda + \mu) = 2\mu p_2$$
$$\lambda(p_{1A} + p_{1B}) = 3\mu p_2$$
……
$$\lambda p_{n-1} = 3\mu p_n \qquad n \geq 3$$

(c) State Probabilities as a function of p_2

Defining $\rho = \frac{\lambda}{\mu}$ and solving the above equations, we get -

$$p_{1B} = \frac{2}{1+\rho} p_2 \qquad p_{1A} = \frac{3+\rho}{\rho(1+\rho)} p_2$$
$$p_0 = \frac{2(3+2\rho)}{\rho^2(1+\rho)} p_2$$
$$p_n = \left(\frac{\lambda}{3\mu}\right)^{n-2} p_2 = \left(\frac{\rho}{3}\right)^{n-2} p_2 \qquad n \geq 2$$

The last expression holds only if $\lambda < 3\mu$. This is the condition for the queue to be stable

(d) The **Normalization Condition** will be the following

$$p_2 \left[\frac{2}{1+\rho} + \frac{3+\rho}{\rho(1+\rho)} + \frac{6+4\rho}{\rho^2(1+\rho)} + \frac{3}{3-\rho} \right] = 1$$

(e) Bonus for Teller A: (Justification) Suppose P_A is the probability that Teller A is working and P_B that Teller B is working. Then taking into account the fact that Teller A works at twice the speed of Teller B, their bonus payments should be in the ratio $2P_A:P_B$.

$$P_A = p_{1A} + \sum_{n=2}^{\infty} p_n = \left[\frac{3+\rho}{\rho(1+\rho)} + \frac{3}{3-\rho} \right] p_2 = \frac{9+3\rho+2\rho^2}{\rho(1+\rho)(3-\rho)} p_2$$

$$P_B = p_{1B} + \sum_{n=2}^{\infty} p_n = \left[\frac{2}{1+\rho} + \frac{3}{3-\rho} \right] p_2 = \frac{9+\rho}{(1+\rho)(3-\rho)} p_2$$

Therefore Teller A should be paid a bonus amount of –

$$2Z \frac{\left(\frac{9+3\rho+2\rho^2}{\rho(1+\rho)(3-\rho)} \right)}{\left(\frac{9+\rho}{(1+\rho)(3-\rho)} \right)} = 2Z \left[\frac{9+3\rho+2\rho^2}{\rho(9+\rho)} \right]$$