Quiz – 2 (EE633 Queueing Systems) 2015-16F
Solution

(a) For Class 2 jobs, the queue will behave like a simple M/G/1/2 queue and the state probabilities can be found accordingly. We then have the following transition probabilities at the departure instants –

$$p_{d,00} = \alpha_0 = L_{B2}(\lambda_2) \qquad p_{d,01} = 1 - L_{B2}(\lambda_2)$$
$$p_{d,10} = L_{B2}(\lambda_2) \qquad p_{d,11} = 1 - L_{B2}(\lambda_2)$$

Balance Equation: $p_{d0} = p_{d0}L_{B2}(\lambda_2) + p_{d1}L_{B2}(\lambda_2) \quad \Rightarrow \quad p_{d1} = p_{d0}\dfrac{[1-L_{B2}(\lambda_2)]}{L_{B2}(\lambda_2)}$

Normalization Condition: $p_{d0} + p_{d1} = 1$

Therefore, state probabilities of the Class 2 customers at the departure instants of the Class 2 customers are - $p_{d0} = L_{B2}(\lambda_2) \qquad p_{d1} = 1 - L_{B2}(\lambda_2)$

Traffic actually offered to the queue $\rho_{c2} = \rho_2(1-P_{B2})$ where $\rho_2 = \lambda_2 \overline{X_2}$

Therefore, $p_0 = 1 - \rho_2(1-P_{B2}) = (1-P_{B2})p_{d0} \quad \Rightarrow \quad 1 - P_{B2} = \dfrac{1}{p_{d0}+\rho_2} = \dfrac{1}{L_{B2}(\lambda_2)+\rho_2}$

Using these,

$$p_0 = (1-P_{B2})p_{d0} = \frac{L_{B2}(\lambda_2)}{\rho_2 + L_{B2}(\lambda_2)}$$
$$p_1 = (1-P_{B2})p_{d1} = \frac{1-L_{B2}(\lambda_2)}{\rho_2 + L_{B2}(\lambda_2)} \qquad \textbf{[2]}$$
$$p_2 = P_{B2} = \frac{\rho_2 + L_{B2}(\lambda_2) - 1}{\rho_2 + L_{B2}(\lambda_2)}$$

will be the required state probabilities at an arbitrary time instant

(b) The queue would be stable for Class 1 customers if the following condition holds.

$$\lambda_1\overline{X_1} + \lambda_2(1-P_{B2})\overline{X_2} < 1 \quad \Rightarrow \quad \rho_1 < \frac{L_{B2}(\lambda_2)}{L_{B2}(\lambda_2)+\rho_2} \qquad \textbf{[1]}$$

(c) Let α = P{no Class 2 arrivals in a Class 2 service time} = $\int_0^\infty e^{-\lambda_2 x} b_2(x)dx = L_{B2}(\lambda_2)$

Therefore $\overline{BP_2} = \sum_{j=1}^{\infty} j\overline{X_2}(1-\alpha)^{j-1}\alpha = \dfrac{\overline{X_2}}{\alpha} = \dfrac{\overline{X_2}}{L_{B2}(\lambda_2)}$ **[1]**

$$L_{BP2}(s) = E\{e^{-s(BP2)}\} = \sum_{n=1}^{\infty} L_{B2}^n(s)(1-\alpha)^{n-1}\alpha = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{(1-\alpha)L_{B2}(s)}{1-(1-\alpha)L_{B2}(s)}\right)$$
$$= \frac{L_{B2}(\lambda_2)L_{B2}(s)}{[1-L_{B2}(s)] + L_{B2}(s)L_{B2}(\lambda_2)} \qquad \textbf{[2]}$$

(d) $$\overline{T} = \overline{X_1} + \lambda_2 \overline{X_1}(\overline{BP2}) = \overline{X_1}\left(1 + \frac{\rho_2}{L_{B2}(\lambda_2)}\right) = \overline{X_1}\left(\frac{L_{B2}(\lambda_2) + \rho_2}{L_{B2}(\lambda_2)}\right)$$ **[1]**

$$L_T(s) = E\{e^{-sT}\} = \sum_{n=0}^{\infty} E\left\{e^{-sX_1} L_{BP2}^n(s) \frac{(\lambda_2 X_1)^n}{n!} e^{-\lambda_2 X_1}\right\}$$

$$= E\left\{\sum_{n=0}^{\infty} e^{-(s+\lambda_2)X_1} \frac{[\lambda_2 X_1 L_{BP2}(s)]^n}{n!}\right\}$$ **[2]**

$$= E\left\{e^{-[s+\lambda_2-\lambda_2 L_{BP2}(s)]X_1}\right\}$$

$$= L_{B1}(s + \lambda_2 - \lambda_2 L_{BP2}(s))$$

Note that you can also obtain $\overline{T}$ from $L_T(s)$

(e) In order to use the standard M/G/1 result, note that $\rho = \lambda_1 \overline{T_1} = \rho_1\left(\frac{L_{B2}(\lambda_2) + \rho_2}{L_{B2}(\lambda_2)}\right)$ and

$$A(z) = L_T(\lambda_1 - \lambda_1 z) = L_{B1}(\lambda_1 - \lambda_1 z + \lambda_2 - \lambda_2 L_{BP2}(\lambda_1 - \lambda_1 z))$$

Therefore,

$$P_1(z) = \frac{(1-\rho)(1-z)L_{B1}(\lambda_1 - \lambda_1 z + \lambda_2 - \lambda_2 L_{BP2}(\lambda_1 - \lambda_1 z))}{L_{B1}(\lambda_1 - \lambda_1 z + \lambda_2 - \lambda_2 L_{BP2}(\lambda_1 - \lambda_1 z)) - z}$$ **[1]**