EE633 Queueing Systems (2015-16F)
Solutions to the Mid-Term Examination

1. (a) $L_B(s) = \left(\frac{\mu}{s+\mu}\right)(0.5) + \left(\frac{\mu}{s+\mu}\right)(0.5)\left(\frac{\mu}{s+\mu}\right)(0.5) + \left(\frac{\mu}{s+\mu}\right)(0.5)\left(\frac{\mu}{s+\mu}\right)(0.5)L_B(s)$

$$L_B(s)\left[1 - \frac{0.25\mu^2}{(s+\mu)^2}\right] = \left(\frac{0.5\mu}{s+\mu}\right)\left[1 + \left(\frac{0.5\mu}{s+\mu}\right)\right]$$

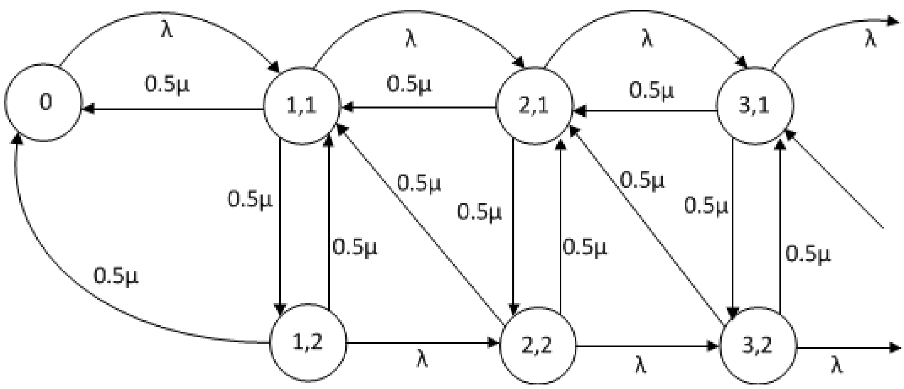
Therefore, $L_B(s) = \frac{0.5\mu(s+1.5\mu)}{s^2 + 2s\mu + 0.75\mu^2} = \frac{0.5\mu}{s+0.5\mu}$

(b) As in (a), we can write $\bar{X} = 0.5\left(\frac{1}{\mu}\right) + 0.25\left(\frac{2}{\mu}\right) + 0.25\left(\frac{2}{\mu} + \bar{X}\right)$

Therefore, $\bar{X} = \frac{2}{\mu}$

(c) The queue will be stable if $\lambda\bar{X} < 1$, i.e. $\frac{\lambda}{\mu} < 0.5$ or $\rho < 0.5$

(d) State Transition Diagram



(e) Traffic offered is 2ρ. Therefore, probability of server being idle will be $(1-2\rho)$

(f) The Balance Equations can be written as follows –

Equation Set 1	Equation Set 2
$p_{1,2}(\lambda+\mu) = p_{1,1}(0.5\mu)$	$\lambda p_0 = 0.5\mu(p_{1,1} + p_{1,2})$
$p_{2,2}(\lambda+\mu) = p_{2,1}(0.5\mu) + p_{1,2}\lambda$	$\lambda(p_{1,1} + p_{1,2}) = 0.5\mu(p_{2,1} + p_{2,2})$
$p_{3,2}(\lambda+\mu) = p_{3,1}(0.5\mu) + p_{2,2}\lambda$	$\lambda(p_{2,1} + p_{2,2}) = 0.5\mu(p_{3,1} + p_{3,2})$
$p_{4,2}(\lambda+\mu) = p_{4,1}(0.5\mu) + p_{3,2}\lambda$	$\lambda(p_{3,1} + p_{3,2}) = 0.5\mu(p_{4,1} + p_{4,2})$
....	....
....	....

From Equation Set 1, we get –

$$(\lambda+\mu)P_2(z)=0.5\mu P_1(z)+\lambda z P_2(z) \qquad \Rightarrow \qquad P_1(z)=2(1+\rho-\rho z)P_2(z) \qquad \textbf{[A]}$$

From Equation Set 2, we get –

$$\lambda\left(p_0+P_1(z)+P_2(z)\right)=\frac{\mu}{2z}\left(P_1(z)+P_2(z)\right)$$

$$P_1(z)+P_2(z)=\frac{2\rho p_0 z}{1-2\rho z}$$

Note that p_0 can be found using the Normalization Condition $P_1(1)+P_2(1)+p_0=1$

Alternatively, we can directly use the result of part (e) to get $p_0=1-2\rho$

Therefore, $\qquad P_1(z)+P_2(z)=(1-2\rho)\left(\frac{2\rho z}{1-2\rho z}\right) \qquad$ **[B]**

Solving [A] and [B], we get –

$$P_2(z)=\frac{2\rho z(1-2\rho)}{(1-2\rho z)(3+2\rho-2\rho z)}$$

$$P_1(z)=\frac{4\rho z(1-2\rho)(1+\rho-\rho z)}{(1-2\rho z)(3+2\rho-2\rho z)}$$

[g] Let $\tilde{P}(z)$ be the generating function for the number in the system. This can be obtained directly from Equation [B] above as $\tilde{P}(z)=(1-2\rho)\left(\frac{2\rho z}{1-2\rho z}\right)+p_0=\frac{1-2\rho}{1-2\rho z}$

[h] Differentiating $\tilde{P}(z)$ and evaluating at z=1, we get

$$N=\frac{2\rho}{1-2\rho}$$

[i] Applying Little's Result, we get $W=\dfrac{2}{\mu(1-2\rho)}$

But we know from the results obtained for the M/G/1 queue that

$$W=\overline{X}+\frac{\lambda\overline{X^2}}{2(1-\lambda\overline{X})}$$

Substituting appropriately from the results obtained earlier, we get

$$\frac{2}{\mu(1-2\rho)}=\frac{2}{\mu}+\frac{\lambda\overline{X^2}}{2(1-2\rho)}$$

Therefore $\qquad \overline{X^2}=\dfrac{8}{\mu^2}$

[Differentiating $L_B(s)$ twice and evaluating it at s=0 gives the same result as above!]

2. (a) Residual Life Approach

P{first service is exceptional}= $e^{-\lambda T}$

$$\overline{BP} = e^{-\lambda T}\left[\overline{X^*} + (\lambda \overline{X^*})\frac{\overline{X}}{1-\lambda\overline{X}}\right] + (1-e^{-\lambda T})\left[\frac{\overline{X}}{1-\lambda\overline{X}}\right]$$

$$= \frac{\overline{X}}{1-\lambda\overline{X}} + e^{-\lambda T}\left[\frac{\overline{X^*}-\overline{X}}{1-\lambda\overline{X}}\right] = \frac{\overline{X}+\overline{\Delta}e^{-\lambda T}}{1-\lambda\overline{X}}$$

Average Length of Busy Period

$$\overline{T_{cycle}} = \frac{1}{\lambda} + \frac{\overline{X}}{1-\lambda\overline{X}} + e^{-\lambda T}\left[\frac{\overline{X^*}-\overline{X}}{1-\lambda\overline{X}}\right] = \frac{1}{\lambda(1-\lambda\overline{X})} + e^{-\lambda T}\left[\frac{\overline{X^*}-\overline{X}}{1-\lambda\overline{X}}\right]$$

Average Cycle Length

$$= \frac{1+\lambda\overline{\Delta}e^{-\lambda T}}{\lambda(1-\lambda\overline{X})} \quad \text{where} \quad \overline{\Delta} = \overline{X^*} - \overline{X}$$

Note that if you consider a time interval of length *t, t→∞*, then on the average it would have *M=λt* arrivals and $\frac{t}{\overline{T_{cycle}}}$ Idle-Busy cycles of which $N = e^{-\lambda T}\frac{t}{\overline{T_{cycle}}}$ are ones with exceptional first service

Mean Residual Time

$$R = \lim_{t\to\infty}\frac{1}{t}\left[\sum_{i=1}^{M-N}\frac{1}{2}X_i^2 + \sum_{j=1}^{N}\frac{1}{2}X_j^{*2}\right] = \lim_{t\to\infty}\frac{1}{2}\left[\frac{M-N}{t}\left(\frac{1}{M-N}\sum_{i=1}^{M-N}X_i^2\right) + \frac{N}{t}\left(\frac{1}{N}\sum_{j=1}^{N}X_j^{*2}\right)\right]$$

$$= \frac{1}{2}\left[\lambda\overline{X^2} + \frac{\lambda e^{-\lambda T}(1-\lambda\overline{X})}{1+\lambda\overline{\Delta}e^{-\lambda T}}\left(\overline{X^{*2}} - \overline{X^2}\right)\right]$$

$$= \frac{1}{2}\lambda\overline{X^2}\left[1 + \frac{e^{-\lambda T}(1-\lambda\overline{X})}{1+\lambda\overline{\Delta}e^{-\lambda T}}\frac{\left(\overline{X^{*2}} - \overline{X^2}\right)}{\overline{X^2}}\right]$$

Mean Waiting Time in Queue

$$W_q = \frac{R}{1-\lambda\overline{X}} = \frac{1}{2}\frac{\lambda\overline{X^2}}{(1-\lambda\overline{X})}\left[1 + \frac{e^{-\lambda T}(1-\lambda\overline{X})}{1+\lambda\overline{\Delta}e^{-\lambda T}}\frac{\left(\overline{X^{*2}} - \overline{X^2}\right)}{\overline{X^2}}\right]$$

This was not asked but you can also easily find the probability p_0 of the server being idle (or the system being empty) as $p_0 = \frac{1/\lambda}{\overline{T_{cycle}}} = \frac{1-\lambda\overline{X}}{1+\lambda\overline{\Delta}e^{-\lambda T}}$. This would be useful to verify the results that you will get in part (b).

(b) Imbedded Markov Chain Approach (PASTA and Kleinrock's Principle are both applicable)

Following the usual notation, the state of the system at the departure instants will be -

$$\begin{aligned} n_{i+1} &= a_{i+1} && n_i = 0, \text{probability} = (1-e^{-\lambda T}), \text{ idle period less than or equal to } T \\ &= a_{i+1}^* && n_i = 0, \text{probability} = e^{-\lambda T}, \text{ idle period more than } T \\ &= n_i + a_{i+1} - 1 && n_i \ge 1 \end{aligned}$$

Since the system is considered at equilibrium, the subscripts can be dropped.

Taking expectations of both sides, we get

$$\bar{N} = p_0\left[(1-e^{-\lambda T})\lambda\overline{X} + e^{-\lambda T}\lambda\overline{X^*}\right] + \bar{N} - (1-p_0) + (1-p_0)\lambda\overline{X}$$

$$p_0\left[(1-\lambda\overline{X}) + \left[(1-e^{-\lambda T})\lambda\overline{X} + e^{-\lambda T}\lambda\overline{X^*}\right]\right] = 1-\lambda\overline{X}$$

$$p_0 = \frac{1-\lambda\overline{X}}{1+\lambda\overline{\Delta}e^{-\lambda T}}$$ *Same as before! Confirms result*

Raising both sides of the equation to the power of *z* and taking expectations, we get the following, where $A(z) = L_B(\lambda-\lambda z)$ and $A^*(z) = L_B^*(\lambda-\lambda z)$

$$P(z) = A(z)\sum_{n=1}^{\infty} p_n z^{n-1} + p_0\left[e^{-\lambda T}A^*(z) + (1-e^{-\lambda T})A(z)\right]$$

$$= \frac{A(z)}{z}\left[P(z)-p_0\right] + p_0 A(z) + p_0 e^{-\lambda T}\left[A^*(z) - A(z)\right]$$

Therefore

$$P(z) = p_0\frac{(1-z)A(z) - ze^{-\lambda T}\{A^*(z)-A(z)\}}{A(z)-z}$$

Note: Answer is incomplete unless p_0 is found somehow! Any method of finding it would be acceptable. Marks will be deducted if it is not found at all!