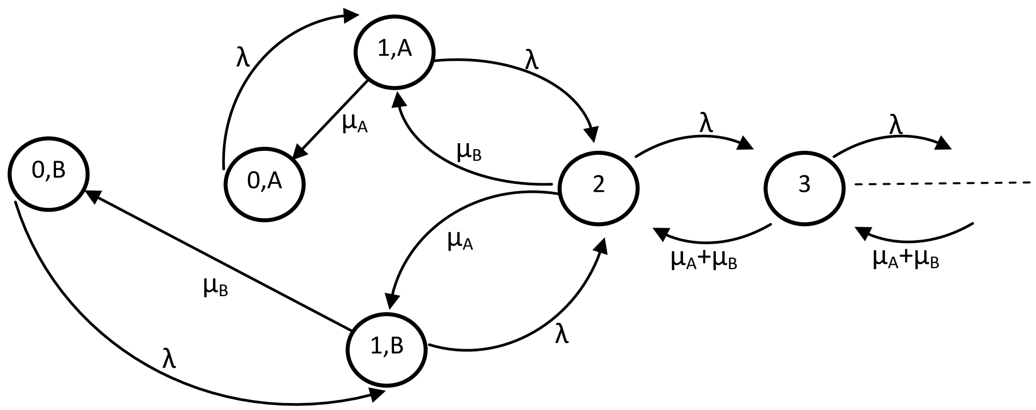
**EE633 2014-2015F**
**Mid Term Exam Solutions**

**1.** **(a)** A State Transition Diagram for this system is given below. Note that states 2, 3,...∞ are normally defined. The other states are defined as follows –

{1,A} one customer in the system, Server A working
{1,B} one customer in the system, Server B working
{0,A} system empty, Server A idle for less time than Server B
{0,B} system empty, Server B idle for less time than Server A




**(b)**

$$p_{1B} = \frac{\lambda}{\mu_B} p_{0B} \qquad p_{1A} = \frac{\lambda}{\mu_A} p_{0A} \qquad p_n = \left(\frac{\lambda}{\mu_A + \mu_B}\right)^{n-2} p_2 \qquad n = 2,3,4,.......$$

$$p_2 = \frac{\lambda}{\mu_A + \mu_B}\left(p_{1A} + p_{1B}\right) \qquad \sum_{n=2}^{\infty} p_n = \frac{\mu_A + \mu_B}{\mu_A + \mu_B - \lambda} p_2$$

$$(\lambda + \mu_A)p_{1A} = \mu_B p_2 + p_{0A}\lambda \qquad (\lambda + \mu_B)p_{1B} = \mu_A p_2 + p_{0B}\lambda$$

$$p_{1A} = \frac{\mu_B}{\lambda} p_2 \qquad p_{0A} = \frac{\mu_A\mu_B}{\lambda^2} p_2 \qquad p_{1B} = \frac{\mu_A}{\lambda} p_2 \qquad p_{0B} = \frac{\mu_A\mu_B}{\lambda^2} p_2$$

$$p_n = \left(\frac{\lambda}{\mu_A + \mu_B}\right)^{n-2} p_2 \qquad n = 2,3,4,....$$

with $$p_2 = \frac{1}{\left(2\frac{\mu_A\mu_B}{\lambda^2} + \frac{\mu_A + \mu_B}{\lambda} + \frac{\mu_A + \mu_B}{\mu_A + \mu_B - \lambda}\right)}$$

**(c)**

P{server A not working} = $p_{0A} + p_{0B} + p_{1B} = \left(\frac{2\mu_A\mu_B}{\lambda^2} + \frac{\mu_A}{\lambda}\right)p_2$

P{server B not working} = $p_{0A} + p_{0B} + p_{1A} = \left(\frac{2\mu_A\mu_B}{\lambda^2} + \frac{\mu_B}{\lambda}\right)p_2$

$$\frac{Penalty\ to\ Server\ A}{Penalty\ to\ Server\ B} = \frac{\lambda\mu_A + 2\mu_B\mu_A}{\lambda\mu_B + 2\mu_A\mu_B}$$

**2 (a)** $\bar{X} = 0.5\left(\frac{2}{\mu}\right) + 0.5\left(\frac{2}{\mu} + \bar{X}\right) \Rightarrow \boxed{\bar{X} = \frac{4}{\mu}}$ $\boxed{p_0 = 1 - \lambda\bar{X} = 1 - 4\rho}$

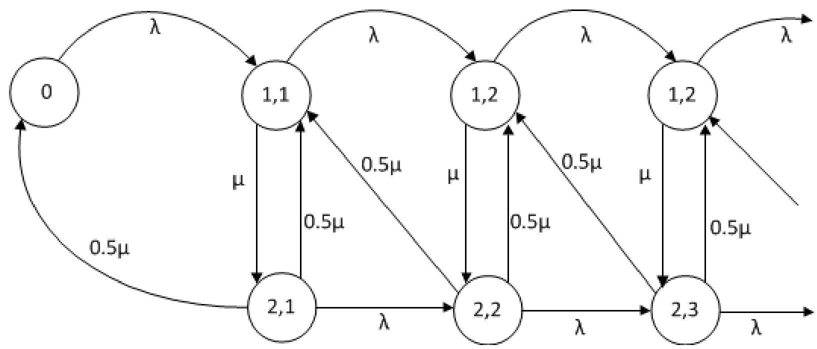
or (longer method!)

$$L_B(s) = 0.5\frac{\mu^2}{(s+\mu)^2} + 0.5\frac{\mu^2}{(s+\mu)^2}L_B(s) \Rightarrow L_B(s) = \frac{\mu^2}{2s^2 + 4s\mu + \mu^2}$$

$$\bar{X} = -L_B'(s)\big|_{s=0} = \frac{4}{\mu}$$

Note that this system will be stable only if $\frac{4\lambda}{\mu} < 1$ or $\rho < 0.25$

**(b)** State Transition Diagram




**(c)** Using $p_0 = 1 - 4\rho$

$\lambda p_0 = 0.5\mu p_{21}$ $\boxed{p_{21} = 2\rho p_0 = 2\rho(1-4\rho)}$

$(\lambda+\mu)p_{11} = 0.5\mu(p_{21} + p_{22}) + \lambda p_0$ $(\lambda+\mu)p_{21} = \mu p_{11}$ $\times z^1$

$(\lambda+\mu)p_{12} = 0.5\mu(p_{22} + p_{23}) + \lambda p_{11}$ $(\lambda+\mu)p_{22} = \mu p_{12} + \lambda p_{21}$ $\times z^2$

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$p_{11} = (1+\rho)p_{21} = 2\rho(1+\rho)p_0$ $\boxed{p_{11} = 2\rho(1+\rho)(1-4\rho)}$

$p_{22} = 2(1+\rho)p_{11} - p_{21} - 2\rho p_0$ $\boxed{p_{22} = 4\rho^2(\rho+2)(1-4\rho)}$

$p_{12} = (1+\rho)p_{22} - \rho p_{21}$ $\boxed{p_{12} = 2\rho^2(2\rho^2 + 6\rho + 3)(1-4\rho)}$

**(d)** Using the balance equations as in (c) (for the equations shown on the right part) and multiplying by $z^i$ as shown and adding LHS and RHS, we get –

$$(\lambda+\mu)P_2(z) = \mu P_1(z) + \lambda z P_2(z)$$

$$(1+\rho-\rho z)P_2(z) = P_1(z) \qquad \boxed{P_2(z) = \frac{P_1(z)}{1+\rho-\rho z}}$$

**(e)** Similarly, using the equations in (c) on the left part, we get –

$$(\lambda+\mu)P_1(z) = 0.5\mu P_2(z) + \frac{0.5\mu}{z}\left[P_2(z) - zp_{21}\right] + \lambda z P_1(z) + \lambda z p_0$$

$$[2(1+\rho)z-2\rho z^2]P_1(z)=(z+1)P_2(z)-zp_{21}+2\rho z^2 p_0$$
$$2\left[1+\rho-\rho z\right]P_1(z)=(z+1)P_2(z)+2\rho z(z-1)p_0$$

$$\boxed{P_1(z)=\frac{(z+1)P_2(z)+2\rho z(z-1)(1-4\rho)}{2(1+\rho-\rho z)}}$$

**3. (a)** $\overline{BP^*}=(\bar{X}+\Delta)+\lambda(\bar{X}+\Delta)\dfrac{\bar{X}}{1-\lambda\bar{X}}=\dfrac{\bar{X}+\Delta}{1-\lambda\bar{X}}$

**(b)** Note that we can use this to get $\overline{T_{cycle}}=\dfrac{1}{\lambda}+\dfrac{\bar{X}+\Delta}{1-\lambda\bar{X}}=\dfrac{1+\lambda\Delta}{\lambda(1-\lambda\bar{X})}$

Therefore, $p_0=\dfrac{1-\lambda\bar{X}}{1+\lambda\Delta}$

**(c)** Since state transition diagram cannot be drawn, the state probabilities will have to be found using the *Imbedded Markov Chain* approach. Using this, for the departure instants, we can write

$$\begin{aligned} n_{i+1} &= a^*_{i+1} \qquad & n_i &= 0 \\ &= n_i + a_{i+1} - 1 \qquad & n_i &= 1,2,3.......\infty \end{aligned}$$

Note that $a_{i+1}$ is the number arriving in a normal service time *X* whereas $a^*_{i+1}$ is the number arriving in the first service time of a busy period, i.e. of length *X*+Δ. The normal service time has L.T. of its pdf as $L_B(s)=\dfrac{1}{s+\mu}$ and L.T. of the pdf of the exceptional first service time is –

$$L_{B^*}(s)=E\{e^{-s(X+\Delta)}\}=\frac{\mu e^{-s\Delta}}{s+\mu}$$

Therefore, $A(z)=L_B(\lambda-\lambda z)=\dfrac{\mu}{\lambda+\mu-\lambda z}$ and $A^*(z)=L_{B^*}(\lambda-\lambda z)=\dfrac{\mu e^{-(\lambda-\lambda z)\Delta}}{\lambda+\mu-\lambda z}$

At equilibrium $P(z)=E\{z^n\}=p_0A^*(z)+A(z)\sum_{n=1}^{\infty}z^{n-1}p_n=p_0A^*(z)+\dfrac{A(z)}{z}\left[P(z)-p_0\right]$

Therefore
$$\begin{aligned} \left(z-A(z)\right)P(z) &= p_0\left[zA^*(z)-A(z)\right] \\ P(z) &= \left(\frac{1-\lambda\bar{X}}{1+\lambda\Delta}\right)\left[\frac{zA^*(z)-A(z)}{z-A(z)}\right]=\left(\frac{1-\lambda\bar{X}}{1+\lambda\Delta}\right)\left[\frac{A(z)}{z-A(z)}\right]\left(ze^{-(\lambda-\lambda z)\Delta}-1\right) \\ &= \left(\frac{1-\lambda\bar{X}}{1+\lambda\Delta}\right)\frac{\mu\left(ze^{-(\lambda-\lambda z)\Delta}-1\right)}{\left(z(\lambda+\mu-\lambda z)-\mu\right)} \end{aligned}$$

**(d)** As per the instruction given, this cannot be done from Residual Life arguments. You have to use the generating function of (c) for this. However, if time permits, you can use The Residual Life approach to cross check your results.

$$\left(z - A(z)\right)P(z) = p_0\left[zA^*(z) - A(z)\right]$$

Differentiating once $\quad (z-A)P' + (1-A')P = p_0[zA^{*\prime} + A^* - A']$

Differentiating again $\quad (z-A)P'' + 2(1-A')P' - A''P = p_0[zA^{*\prime\prime} + 2A^{*\prime} - A'']$

Evaluating at $z$=1, using $A' = \lambda\bar{X}, A^{*\prime} = \lambda(\bar{X}+\Delta), A'' = \lambda^2\overline{X^2}, A^{*\prime\prime} = \lambda^2(\overline{X^2} + 2\Delta\bar{X} + \Delta^2)$, we get

$$2(1-\lambda\bar{X})N - \lambda^2\overline{X^2} = p_0[\lambda^2(\overline{X^2} + 2\Delta\bar{X} + \Delta^2) + 2\lambda\bar{X} - \lambda^2\overline{X^2}]$$

$$= p_0[2\lambda\bar{X}(1+\lambda\Delta) + \lambda^2\Delta^2]$$

$$\boxed{N = \frac{\lambda^2\overline{X^2}}{2(1-\lambda\bar{X})} + \frac{2\lambda\bar{X}(1+\lambda\Delta) + \lambda^2\Delta^2}{2(1+\lambda\Delta)}}$$