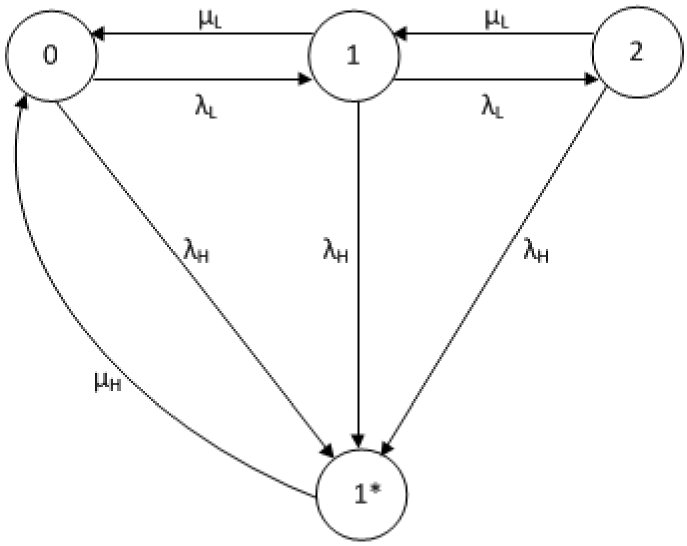
EE 633, Queueing Systems
Mid-Term Examination (2012-2013S)
Solutions

1. (a) State Transition Diagram - States 1 and 2 are for LP jobs in system and 1* is for one HP job in the system **[4]**



(b) $\lambda_H(p_0 + p_1 + p_2) = \mu_H p_{1^*}$

Since $(p_0 + p_1 + p_2) = 1 - p_{1^*}$ we get $p_{1^*} = \dfrac{\lambda_H}{\mu_H + \lambda_H}$

Therefore, P{system has no HP customers} = $1 - p_{1^*} = \dfrac{\mu_H}{\mu_H + \lambda_H}$ **[2]**

(c) $p_2(\lambda_H + \mu_L) = p_1\lambda_L$ and $p_1(\lambda_H + \lambda_L + \mu_L) = p_0\lambda_L + p_2\mu_L$

Solving - $p_1 = p_0 \dfrac{\lambda_L(\lambda_H + \mu_L)}{(\lambda_H^2 + 2\lambda_H\mu_L + \mu_L^2 + \lambda_L\lambda_H)}$ $p_2 = p_0 \dfrac{\lambda_L^2}{(\lambda_H^2 + 2\lambda_H\mu_L + \mu_L^2 + \lambda_L\lambda_H)}$

Using the Normalization Constraint $(p_0 + p_1 + p_2) = 1 - p_{1^*} = \dfrac{\mu_H}{\mu_H + \lambda_H}$

Therefore, $p_0 \dfrac{\lambda_H^2 + 2\lambda_H\mu_L + \mu_L^2 + \lambda_L\lambda_H + \lambda_L(\lambda_H + \mu_L) + \lambda_L^2}{(\lambda_H^2 + 2\lambda_H\mu_L + \mu_L^2 + \lambda_L\lambda_H)} = \dfrac{\mu_H}{\lambda_H + \mu_H}$

$$p_0 = \frac{(\lambda_H^2 + 2\lambda_H\mu_L + \mu_L^2 + \lambda_L\lambda_H)\mu_H}{(\lambda_H + \mu_H)(\lambda_H^2 + 2\lambda_H\mu_L + \mu_L^2 + \lambda_L\lambda_H + \lambda_L(\lambda_H + \mu_L) + \lambda_L^2)}$$
$$= \frac{(\lambda_H^2 + 2\lambda_H\mu_L + \mu_L^2 + \lambda_L\lambda_H)\mu_H}{(\lambda_H + \mu_H)\left[(\lambda_H + \mu_L + \lambda_L)^2 - \lambda_L\mu_L\right]}$$

[5]

2

(a)

$$L_B(s) = 0.5L_{B1}(s) + 0.5L_{B2}(s) = 0.5L_{B1}(s) + 0.5\frac{\mu}{s+\mu}L_{B1}(s)$$

$$= 0.5L_{B1}(s)\frac{s+2\mu}{s+\mu}$$

where

$$L_{B1}(s) = 0.5\left(\frac{\mu}{s+\mu}\right)\sum_{i=0}^{\infty}\left[0.5\left(\frac{\mu}{s+\mu}\right)\left(\frac{\mu}{s+\mu}\right)\right]^i$$

$$= \frac{\mu(s+\mu)}{(2s^2 + 4\mu s + \mu^2)}$$

Alternative approach - $L_{B1}(s) = 0.5\left(\frac{\mu}{s+\mu}\right) + 0.5\left(\frac{\mu}{s+\mu}\right)^2 L_{B1}(s)$

Therefore $L_{B1}(s) = \frac{\mu(s+\mu)}{(2s^2 + 4\mu s + \mu^2)}$ as before

Therefore, LT of the pdf of the overall service time is $L_B(s) = \frac{0.5\mu(s+2\mu)}{(2s^2 + 4\mu s + \mu^2)}$ **[5]**

(b) $\bar{X} = -L'_B(s)\big|_{s=0}$

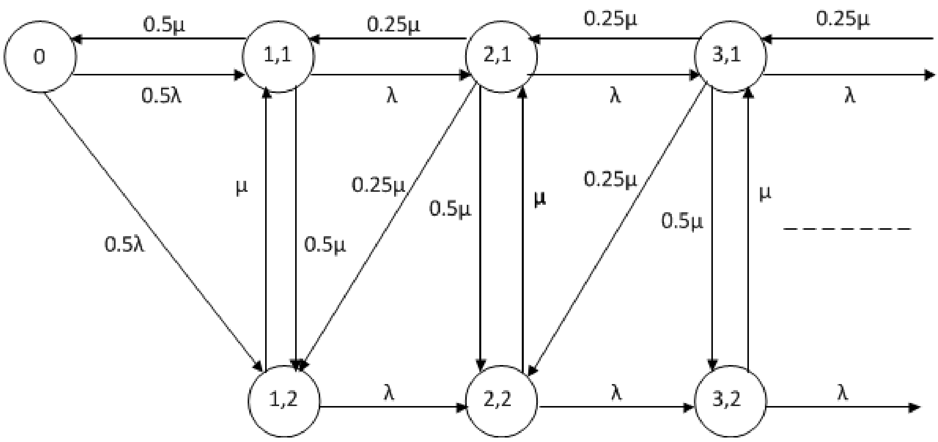
$$L'_B(s) = \frac{0.5\mu}{(2s^2 + 4\mu s + \mu^2)} - \frac{0.5\mu(s+2\mu)(4s+4\mu)}{(2s^2 + 4\mu s + \mu^2)^2}$$

$$= -\frac{0.5\mu(2s^2 + 8\mu s + 7\mu^2)}{(2s^2 + 4\mu s + \mu^2)^2}$$

Therefore $\bar{X} = \frac{3.5}{\mu}$

The M/-/1 queue will be stable if $\lambda\bar{X} < 1$ or $\frac{\lambda}{\mu} < 3.5^{-1} = 0.286$ **[3]**

(c) State Transition Diagram with the usual definition of system states



[5]

3.

(a) $\bar{I}$ **=** Mean Idle Period in a cycle

$$\bar{I} = p\left[\bar{V} + \int_0^\infty e^{-\lambda v} f_V(v)\frac{1}{\lambda}dv\right] + (1-p)\frac{1}{\lambda} = \frac{1}{\lambda} + p\frac{\lambda\bar{V} + L_V(\lambda) - 1}{\lambda}$$
$$= \frac{1 + p\left(\lambda\bar{V} + L_V(\lambda) - 1\right)}{\lambda}$$

We can find the mean Busy Period explicitly to find the Mean Cycle Length, or simply use the fact that $\frac{\bar{I}}{\bar{T}_{cycle}} = (1-\lambda\bar{X})$ to get

$$\bar{T}_{cycle} = \frac{1 + p\left(\lambda\bar{V} + L_V(\lambda) - 1\right)}{\lambda(1-\lambda\bar{X})}$$

Therefore $$P(V) = \frac{p\bar{V}}{\bar{T}_{cycle}} = \frac{p\lambda\bar{V}(1-\lambda\bar{X})}{1 + p\left(\lambda\bar{V} + L_V(\lambda) - 1\right)}$$ **[5]**

Note: To find the Mean Busy Period explicitly, we can do the following -

$$\overline{BP} = (1-p)\left(\frac{\bar{X}}{1-\lambda\bar{X}}\right) + p\left[\sum_{n=1}^{\infty}\int_0^\infty \frac{(\lambda v)^n}{n!}e^{-\lambda v} f_V(v)\left(\frac{n\bar{X}}{1-\lambda\bar{X}}\right)dv + \int_0^\infty e^{-\lambda v} f_V(v)\left(\frac{\bar{X}}{1-\lambda\bar{X}}\right)dv\right]$$
$$= \left(\frac{\bar{X}}{1-\lambda\bar{X}}\right)\left[1 + p\left(\lambda\bar{V} + L_V(\lambda) - 1\right)\right]$$

Using this $\bar{T}_{cycle}$ can be found to give the same result as given earlier.

(b) Consider a large time interval *t* when the queue is in equilibrium. Let R_t be the mean residual time (service or vacation) in this interval. Let *M* be the number of arrivals and *N* the number of vacations in this interval.

$$\lim_{t\to\infty}\left(\frac{M}{t}\right) = \lambda \qquad \lim_{t\to\infty}\left(\frac{N}{t}\right) = \frac{p}{\bar{T}_{cycle}} = \frac{p\lambda(1-\lambda\bar{X})}{1 + p\left(\lambda\bar{V} + L_V(\lambda) - 1\right)}$$
$$R = \lim_{t\to\infty} R_t = \frac{\lambda\overline{X^2}}{2} + \frac{p\lambda(1-\lambda\bar{X})}{1 + p\left(\lambda\bar{V} + L_V(\lambda) - 1\right)}\left(\frac{\overline{V^2}}{2}\right)$$

Therefore $$W_q = \frac{R}{1-\lambda\bar{X}} = \frac{\lambda\overline{X^2}}{2(1-\lambda\bar{X})} + \frac{p\lambda}{1 + p\left(\lambda\bar{V} + L_V(\lambda) - 1\right)}\left(\frac{\overline{V^2}}{2}\right)$$

(**Note:** The case *p*=1 corresponds to the standard single vacation M/G/1 queue)

Therefore $$N_q = \lambda W_q = \frac{\lambda^2\overline{X^2}}{2(1-\lambda\bar{X})} + \frac{p\lambda^2}{1 + p\left(\lambda\bar{V} + L_V(\lambda) - 1\right)}\left(\frac{\overline{V^2}}{2}\right)$$ **[5]**

(c) Let NV be the event of no vacation (probability (1-p)) in a cycle and NV* the event of one vacation in a cycle. If there is a vacation, then let j (j≥0) be the number of arrivals in that vacation.

$$\begin{aligned} n_{i+1} &= n_i + a_{i+1} - 1 & n_i &\geq 1 \\ &= a_{i+1} & n_i &= 0, NV \\ &= a_{i+1} & n_i &= 0, NV^*, j = 0 \\ &= a_{i+1} + j - 1 & n_i &= 0, NV^*, j \geq 1 \end{aligned}$$

where

$$f_j = \int_{v=0}^{\infty} \frac{(\lambda v)^j}{j!} e^{-\lambda v} f_V(v)dv$$

$$F(z) = \sum_{j=0}^{\infty} f_j z^j = L_V(\lambda - \lambda z)$$

Taking expectations of LHS and RHS at equilibrium

$$\bar{n} = \lambda \bar{X} + \bar{n} - (1 - p_0) + p_0 p \sum_{j=1}^{\infty} (j-1) f_j$$

$$1 - \lambda \bar{X} = p_0 \left[1 + p\left(\lambda \bar{V} - 1 + L_V(\lambda) \right) \right]$$

$$p_0 = \frac{1 - \lambda \bar{X}}{1 + p\left(\lambda \bar{V} + L_V(\lambda) - 1 \right)}$$

(**Note:** Compare with case p=1 as mentioned before.)

$$\begin{aligned} P(z) &= A(z)\left[\sum_{n=1}^{\infty} z^{n-1} p_n + p_0(1-p) + p_0 p f_0 + p_0 p \sum_{j=1}^{\infty} z^{j-1} f_j \right] \\ &= A(z)\left[\frac{1}{z}\left\{ P(z) - p_0 \right\} + p_0(1-p) + p_0 p L_V(\lambda) + \frac{p_0 p}{z}\left\{ F(z) - L_V(\lambda) \right\} \right] \end{aligned}$$

$$\begin{aligned} \left[z - A(z) \right] P(z) &= p_0 A(z) \left[p\left\{ L_V(\lambda - \lambda z) - (1-z) L_V(\lambda) \right\} - 1 + z(1-p) \right] \\ &= p_0 A(z) \left[p\left\{ L_V(\lambda - \lambda z) - (1-z) L_V(\lambda) \right\} + z(1-p) - 1 \right] \end{aligned}$$

Therefore $$P(z) = p_0 \left(\frac{A(z)}{z - A(z)} \right) \left[p\left\{ L_V(\lambda - \lambda z) - (1-z) L_V(\lambda) \right\} + z(1-p) - 1 \right]$$ **[6]**

with $A(z) = L_B(\lambda - \lambda z)$ and $p_0 = \dfrac{1 - \lambda \bar{X}}{1 + p\left(\lambda \bar{V} + L_V(\lambda) - 1 \right)}$