EE 633, Queueing Systems (2017-18F)
Mid-Term Exam, Solutions

1. Let $\overline{X}$ be the mean service time of the server with $L_B(s)$ as the L.T. of the pdf of the service.

(a) $$\overline{X} = 0.5\left(\frac{0.5}{\mu} + \overline{X}\right) + 0.25\left(2\times\frac{0.5}{\mu}\right) + 0.25\left(2\times\frac{0.5}{\mu} + \overline{X}\right)$$

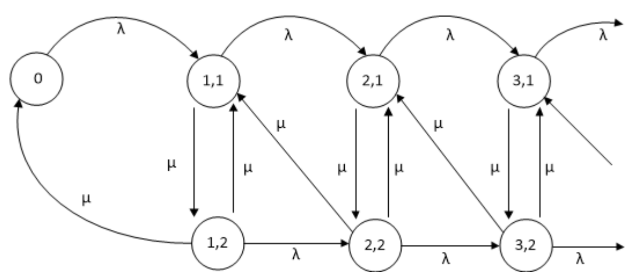
Therefore $\overline{X} = \dfrac{3}{\mu}$ **[5]**

(b) $$L_B(s) = 0.5\left(\frac{2\mu}{s+2\mu}\right)L_B(s) + 0.25\left(\frac{2\mu}{s+2\mu}\right)^2 L_B(s) + 0.25\left(\frac{2\mu}{s+2\mu}\right)^2$$

$$L_B(s)\left[1 - \frac{\mu}{s+2\mu} - \frac{\mu^2}{(s+2\mu)^2}\right] = \frac{\mu^2}{(s+2\mu)^2}$$ **[5]**

$$L_B(s) = \frac{\mu^2}{s^2 + 3\mu s + \mu^2}$$

(c) State Transition Diagram **[5]**



(d) Balance Equations

$$\lambda p_0 = \mu p_{12}$$
$$\lambda(p_{11} + p_{12}) = \mu p_{22} \qquad (\lambda + 2\mu)p_{12} = \mu p_{11}$$
$$\lambda(p_{21} + p_{22}) = \mu p_{32} \qquad (\lambda + 2\mu)p_{22} = \mu p_{21} + \lambda p_{12}$$
$$\lambda(p_{31} + p_{32}) = \mu p_{42} \qquad (\lambda + 2\mu)p_{32} = \mu p_{31} + \lambda p_{22}$$ **[5]**

....

Using the balance equations of (d) (multiply LHS and RHS by z^n and sum), we get

$$\rho\left[P_1(z) + P_2(z)\right] = \frac{1}{z}P_2(z) - p_{12} \qquad (\rho+2)P_2(z) = P_1(z) + \rho z P_2(z)$$ **[5+5]**

$$-\rho z P_1(z) + (1-\rho z)P_2(z) = \rho z p_0 \qquad P_1(z) = \left(2 + \rho(1-z)\right)P_2(z)$$

You do not need to find these but we can get the generating functions as –

$$P_1(z) = p_0 \frac{\rho z(2+\rho-\rho z)}{1-3\rho z-\rho^2 z+\rho^2 z^2} \qquad P_2(z) = p_0 \frac{\rho z}{1-3\rho z-\rho^2 z+\rho^2 z^2}$$

(e) Evaluating for *z*=1, we get –

$$P_1(1) = 2P_2(1) \quad \text{and} \quad P_2(1) = p_0 \frac{\rho}{1-3\rho}$$

which gives $p_0 = 1-3\rho$ **[5]**

The system will be stable for $\rho < 1/3$. **[5]**

We also find that $P_1(1) = 2\rho \quad P_2(1) = \rho$. *These are required later in* **(f).**

(f) Differentiating the equations of **(e)** and solving, we get

$$-\rho P_1(z) - \rho z P_1'(z) + (1-\rho z)P_2'(z) - \rho P_2(z) = \rho p_0$$

$$P_1'(z) = \left[\rho(1-z)+2\right]P_2'(z) - \rho P_2(z)$$

$$-\rho P_1(1) - \rho P_1'(1) + (1-\rho)P_2'(1) - \rho P_2(1) = \rho p_0 = \rho - 3\rho^2$$

$$P_1'(1) = 2P_2'(1) - \rho P_2(1)$$

$$-\rho P_1'(1) + (1-\rho)P_2'(1) = \rho$$

$$-P_1'(1) + 2P_2'(1) = \rho^2$$

$$P_1'(1) = \frac{2\rho-\rho^2+\rho^3}{1-3\rho} \qquad P_2'(1) = \frac{\rho(1-\rho^2)}{1-3\rho}$$

Mean Number in System = $P_1'(1) + P_2'(1) = \dfrac{\rho(3-\rho)}{1-3\rho}$ **[10]**

2. **(a)** For this system, the imbedded Markov Chain at the customer departure instants may be written as (using standard notation) –

$$
\begin{aligned}
n_{i+1} &= a^{*}_{i+1} && n_i = 0,1 \\
&= n_i + a_{i+1} - 1 && n_i \geq 2
\end{aligned}
$$

[5]

Taking expectations of the LHS and RHS of the above equation at equilibrium, and using $\rho = \lambda \overline{X}$ and $\rho^{*} = \lambda \overline{X^{*}}$, we get $N = (p_0 + p_1)\rho^{*} + (N - p_1) + (1 - p_0 - p_1)\rho - (1 - p_0 - p_1)$

This gives $p_0(\rho^{*} - \rho + 1) + p_1(\rho^{*} - \rho) = (1 - \rho)$ **(A)** **[5]**

The generating function $P(z)$ may be found as follows –

$$P(z) = E\{z^n\} = (p_0 + p_1)A^{*}(z) + A(z)\sum_{n=2}^{\infty} p_n z^{n-1}$$

$$zP(z) = (p_0 + p_1)zA^{*}(z) + A(z)\left[P(z) - p_0 - p_1 z\right]$$

Therefore, $$P(z) = \frac{(p_0 + p_1)zA^{*}(z) - (p_0 + p_1 z)A(z)}{z - A(z)}$$ **[10]**

Note that we still need to find p_0 and p_1 to complete the derivation of $P(z)$. We have one relationship between p_0 and p_1 given by (A) and need to find one more. Using the Normalization Condition of $P(z)\big|_{z=1} = 1$ will not be useful. (Why? *It will give you Eq (A) once again!*)
Since p_1 also appears in the expression for $P(z)$, we can try to get another equation by evaluating p_1 from $P(z)$ using $P(0) = p_0$ and $\frac{d}{dz}P(z)\Big|_{z=0} = p_1$.

$$(z - A(z))P(z) = (p_0 + p_1)zA^{*}(z) - (p_0 + p_1 z)A(z)$$

$$
\begin{aligned}
(z - A(z))P'(z) + (1 - A'(z))P(z) &= (p_0 + p_1)A^{*}(z) + (p_0 + p_1)zA^{*\prime}(z) \\
&\quad - (p_0 + p_1 z)A'(z) - p_1 A(z)
\end{aligned}
$$

(B)

Note that

$$A(z) = L_B(\lambda - \lambda z) \quad A(0) = L_B(\lambda) \quad A(1) = 1 \quad A'(1) = -\lambda L_B'(0) = \lambda \overline{X} = \rho$$

$$A^{*}(z) = L_B^{*}(\lambda - \lambda z) \quad A^{*}(0) = L_B^{*}(\lambda) \quad A^{*}(1) = 1 \quad A^{*\prime}(1) = -\lambda L_B^{*}{}'(0) = \lambda \overline{X^{*}} = \rho^{*}$$

Using these and evaluating (B) at *z*=0, we get

$$-p_1 L_B(\lambda) + p_0\left[1 + \lambda L_B'(\lambda)\right] = (p_0 + p_1)L_B^{*}(\lambda) + p_0 \lambda L_B'(\lambda) - p_1 L_B(\lambda)$$

Therefore, $$p_0 = (p_0 + p_1)L_B^{*}(\lambda) \quad \Rightarrow \quad p_1 = p_0 \frac{1 - L_B^{*}(\lambda)}{L_B^{*}(\lambda)}$$ **[C]**

Using [A] and [C] to solve for p_0 and p_1, we get –

$$p_0 = \frac{(1-\rho)L_B^*(\lambda)}{\rho^* - \rho + L_B^*(\lambda)} \qquad p_1 = \frac{(1-\rho)\left[1-L_B^*(\lambda)\right]}{\rho^* - \rho + L_B^*(\lambda)}$$ **[5+5]**

These p_0 and p_1 can now be used in the expression for $P(z)$ given earlier.

Note also that $p_0 + p_1 = \dfrac{p_0}{L_B^*(\lambda)}$. This will be useful later!

(b) Mean Number in System Using the expression for $P(z)$ given earlier, we get –

$$(z-A)P = (p_0 + p_1)zA^* - (p_0 + p_1 z)A$$

Differentiating successively w.r.t *z*,

$$(z-A)P = (p_0 + p_1)zA^* - (p_0 + p_1 z)A$$
$$(z-A)P' + (1-A')P = (p_0 + p_1)A^* + (p_0 + p_1)zA^{*\prime} - (p_0 + p_1 z)A' - p_1 A$$
$$(z-A)P'' + 2(1-A')P' - A''P = 2(p_0 + p_1)A^{*\prime} + (p_0 + p_1)zA^{*\prime\prime} - (p_0 + p_1 z)A'' - 2p_1 A'$$

Substituting *z*=1 in the last line above,

$$2(1-\rho)N - \lambda^2\overline{X^2} = 2(p_0 + p_1)\rho^* + (p_0 + p_1)\lambda^2\overline{X^{*2}} - (p_0 + p_1)\lambda^2\overline{X^2} - 2p_1\rho$$
$$\cdots = (p_0 + p_1)\left[\lambda^2\left(\overline{X^{*2}} - \overline{X^2}\right) + 2\left(\rho^* - \rho\right)\right] + 2p_0\rho$$

$$N = \frac{\lambda^2\overline{X^2}}{2(1-\rho)} + (p_0 + p_1)\left[\frac{\lambda^2\left(\overline{X^{*2}} - \overline{X^2}\right) + 2\left(\rho^* - \rho\right)}{2(1-\rho)}\right] + p_0\frac{\rho}{1-\rho}$$

or $$N = \frac{\lambda^2\overline{X^2}}{2(1-\rho)} + \frac{p_0}{L_B^*(\lambda)}\left[\frac{\lambda^2\left(\overline{X^{*2}} - \overline{X^2}\right) + 2\left(\rho^* - \rho\right)}{2(1-\rho)}\right] + p_0\frac{\rho}{1-\rho}$$ **[15]**

(c) It is easy to verify that when the exceptional service is the same as the regular service, all the above expressions reduce to the expressions for a standard M/G/1 queue

First term is $\dfrac{\lambda^2\overline{X^2}}{2(1-\rho)}$, as usual

Second term is 0

Third term is $\lambda\overline{X}$ since p_0 becomes $1-\rho$ **[5]**