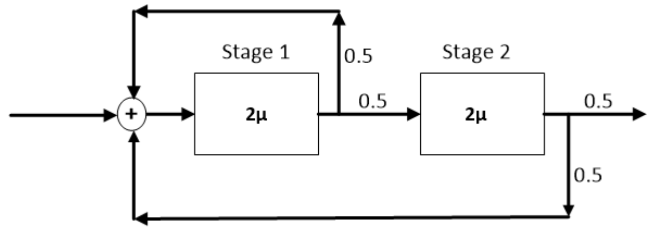
EE 633, Queueing Systems (2017-18F)
Mid-Term Exam

*Maximum Marks: 100 (**will be scaled to 30**)* *Time: 2 hours*
It would be helpful for you to solve the parts of a question in the sequence they are given.
Please read the questions carefully before you start working on them. There are no marks for "misunderstanding" what is being asked! No part marking will be done beyond what is given

1. Consider an **M/M/1** queue where the server provides service in two stages, as shown. The service duration of each stage is exponentially distributed with rate 2μ. **For notational convenience, use** $\rho = \frac{\lambda}{\mu}$ in the following. (Arrival rate is λ.)



Server Model for the M/M/1/∞ Queue
(each stage provides exponentially distributed service times with the rates as shown)

IMPORTANT: Use the Method of Stages to solve this problem. DO NOT USE the analytical approach followed for an M/G/1 queue to do the following.

(a) What is the mean service time for this server? **[5]**
(b) What is distribution of the overall service time? (Giving the L.T. of the pdf will be sufficient!) **[5]**
(c) Draw the **State Transition Diagram** for this queue at equilibrium, representing the system state by (*n*, *j*) where *n* is the number in the system and *j* is the stage in which the customer in-service is currently being served and representing the state where the system is empty as state 0. **[5]**
(d) Write the **Balance Equations** that will be needed to solve for the system state probabilities and use them to find expressions relating the generating functions $P_1(z) = \sum_{n=1}^{\infty} p_{n,1} z^n$ and $P_2(z) = \sum_{n=1}^{\infty} p_{n,2} z^n$ in terms of each other, p_0 and ρ. **[5+5+5]**
(**Note:** We can solve these to get $P_1(z)$ and $P_2(z)$ in terms of p_0 and ρ but you are not asked to do that.)
(e) Use the Normalization Condition and your results of (c) to find p_0. *Use this* to obtain the condition under which the system will be stable. **[5+5]**
(f) Find the Mean Number in Queue (i.e. *N*) **[10]**

2. Consider an M/G/1 queue where the normal service time is a random variable *X* (pdf $b(t)$, cdf $B(t)$ and L.T. of the pdf is $L_B(s)$). This is the kind of service given to the jobs in this queue in **all situations except when the number of jobs in the system is ONE at the time instant when service starts**. When that happens, the service time (**for that job**) is an exceptional one of duration given by the random variable X^* (pdf $b^*(t)$, cdf $B^*(t)$ and L.T. of the pdf is $L_B^*(s)$).
As usual, arrivals to this queue come from a Poisson process with rate λ.

Let $P(z) = \sum_{n=0}^{\infty} p_n z^n$ define the generating function of the system states with $p_n \quad n = 0,1,2,......$ as the equilibrium probability of state n.

A helpful suggestion: As you do each of the following, keep in mind what would happen if X and X had the same distribution. This should help you to crosscheck your work.*

(a) Find $P(z)$ in terms of the moments of *X* and *X** and the L.T. of their pdf's. **[30]**

(b) Find the Mean Number in the System (i.e. *N*) **[15]**

(c) Verify that the expression for (b) reduces to the known result for the standard M/G/1 queue when *X* and *X** have the same distribution. **[5]**

(For notational convenience, use $\rho = \lambda \overline{X}$ *and* $\rho^* = \lambda \overline{X^*}$*.)*

Note: *You can of course follow your own procedure to obtain this. However, for your convenience, the steps recommended are given next along with their associated marks.*

i. Write the equations relating the states at the imbedded points where a Markov Chain can be defined for this system. **[5]**
(*Be careful writing these! These would be slightly different from what you would usually write for a standard M/G/1 queue. In particular be careful about what happens when* $n_i = 0$ *and* $n_i = 1$.)

ii. Take expectations of the LHS and RHS at equilibrium to get a relationship between two state probabilities (that will appear). These are still unknown and will have to be determined later. **[5]**

iii. Obtain $P(z)$ in terms of these (as yet unknown) probabilities and the moments of *X*, *X** and the L.T.s of their pdf's. **[10]**

iv. Use $P(z)$ of (iii) to get a second equation between the two unknown state probabilities (i.e. the ones in (ii)). Solve the two equations to determine these probabilities. **[10]**
Hint: Using standard notation, here are a few things, which might help.

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \quad P(0) = p_0 \quad P'(0) = p_1 \quad P(1) = 1 \quad P'(1) = N$$

$$A(z) = L_B(\lambda - \lambda z) \quad A'(z) = -\lambda L_B'(\lambda - \lambda z) \quad A''(z) = \lambda^2 L_B''(\lambda - \lambda z)$$

$$A(0) = L_B(\lambda) \quad A'(0) = -\lambda L_B'(\lambda) \quad A(1) = 1 \quad A'(1) = \lambda \overline{X} = \rho \quad A''(1) = \lambda^2 \overline{X^2}$$

$$A^*(z) = L_B^*(\lambda - \lambda z) \quad A^{*\prime}(z) = -\lambda L_B^{*\prime}(\lambda - \lambda z) \quad A^{*\prime\prime}(z) = \lambda^2 L_B^{*\prime\prime}(\lambda - \lambda z)$$

$$A^*(0) = L_B^*(\lambda) \quad A^{*\prime}(0) = -\lambda L_B^{*\prime}(\lambda) \quad A^*(1) = 1 \quad A^{*\prime}(1) = \lambda \overline{X^*} = \rho^* \quad A^{*\prime\prime}(1) = \lambda^2 \overline{X^{*2}}$$

This is sufficient to complete the derivation of $P(z)$ *as asked in this problem*. (There is no need to simplify this further by substituting in the earlier expression for $P(z)$!)

v. Use $P(z)$ in the usual fashion to get the mean number in the system for (b). **[15]**

vi. Verify (c) as asked using $\overline{X^*} = \overline{X} \quad \rho^* = \rho \quad \overline{X^{*2}} = \overline{X^2}$ **[5]**