EE 633, Queueing Systems (2016-17F)
Mid-Term Exam

Maximum Marks 50 (25+25: ***will be scaled to 30****)* *Time: 2 hours*
It would be helpful for you to solve the parts of a question in the sequence they are given.
Please read the questions carefully before you start working on them. There are no marks for "misunderstanding" what is being asked!

1. M/M/1 Queue with Conditional Exceptional First Service
Consider a M/M/1 queue where the arrivals come from a Poisson process at rate λ. Normal service in this queue is done at an exponential service rate μ. **Whenever the system becomes empty**, the server first enters an **Active Idle** state for some time, and then, if there are no arrivals during that state, it eventually goes into a **Passive Idle** state. If an arrival comes while the system is in the Active Idle state, then service is resumed at the normal service rate μ and the queue continues operation normally until the next time it becomes idle. However, if there are no arrivals during the Active Idle state and the server has entered the Passive Idle state, then the **first arrival** is served at the exceptional rate μ^*; after that job is served, the queue resumes normal operation once again (i.e. the server resumes working at the normal rate μ) until the system once again becomes empty.
Assume that the server stays in the Active Idle state for an exponentially distributed time with parameter γ, i.e. the time it stays in that state has pdf $\gamma e^{-\gamma t} \quad t \geq 0$**.** For the following, assume that the queue is in equilibrium.

Define the system state as follows -

- n: number in system when service is provided at rate μ
- n^*: number in system when service is provided at rate μ^*
- OA: server idle in Active Idle state
- OP: server idle in Passive Idle state

(a) Draw the State Transition Diagram of the system at equilibrium. **[5]**

(b) Write the balance equations that will be needed to solve for the probabilities of the system state (as defined above). **[4]**

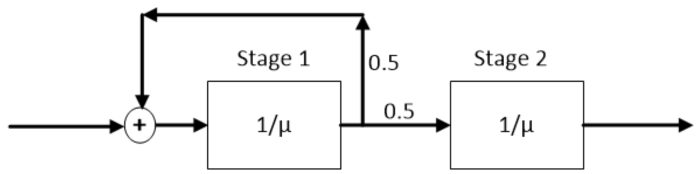
(c) Defining $P(z) = \sum_{n=1}^{\infty} p_n z^n$ and $P^*(z) = \sum_{n=1}^{\infty} p_{n^*} z^n$, give expressions for $P(z)$ and $P^*(z)$ in terms of *z, λ, μ, μ^*, γ* **and** p_{OP}. **[3+3]**
(**Note:** You are **not** being asked to find the probability p_{OP}.)

(d) If we observe the busy periods of the queue over a long time interval and consider their long term average behaviour, then in what fraction of these busy periods will the server be working at rate μ^*? **[4]**

(e) Consider the system when $\gamma=\lambda$. For this system, **use the Busy Period Approach** to find p_0, the probability that there are no jobs in the system. **[6]**

Note: (i) No results from the previous parts (a)-(d) are to be used for (e).
(ii) A good sanity check for your answer would be to see how it fares when $\mu=\mu^*$
(iii) Of course, if you feel brave and have enough time, then you can use the Normalization Condition and the results from (c) to cross-check your answer.
(iv) The result of this part will also allow you to see the condition under which the queue will be in equilibrium for the given value of γ. Is the result obvious? (**Not asked!**)

2. Consider an **M/M/1** queue where the server provides service in two stages, as shown. The service duration of each stage is exponentially distributed with rate μ.



Server Model for the M/M/1/∞ Queue

For notational convenience, use $\rho = \frac{\lambda}{\mu}$

(a) Draw the **State Transition Diagram** for this queue at equilibrium, representing the system state by (n, j) where n is the number in the system and j is the stage in which the customer in-service is currently being served and representing the state where the system is empty as state 0. **[5]**

(b) Write the **Balance Equations** that will be needed to solve for the system state probabilities. **[5]**

(c) Use the balance equations of (c) to give expressions relating the generating functions

$P_1(z) = \sum_{n=1}^{\infty} p_{n,1} z^n$ $P_2(z) = \sum_{n=1}^{\infty} p_{n,2} z^n$ in terms of each other, p_0 and ρ. **[5]**

(**Note:** It should be possible to solve these to get $P_1(z)$ and $P_2(z)$ in terms of p_0 and ρ but you are not being asked to do that.)

(d) Use the appropriate Normalization Condition and your results of (c) to find p_0. **[5]**

(e) Show that the mean number waiting in queue (N_q) will be $\frac{7\rho^2}{(1-3\rho)}$ **(5)**

(**Note:** There will not be any part marking for (e) but the steps must be clearly shown. Show means to really show! No marks for fudging the steps or the result.)