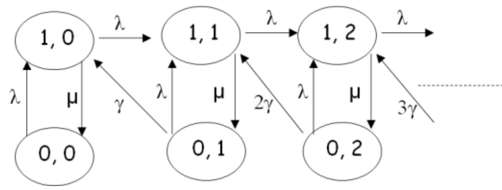
EE633 2014-2015F
Final Exam Solutions

1. (a) State Transition Diagram



Balance Equations

$$(\lambda + n\gamma) p_{0,n} = \mu p_{1,n} \qquad n \geq 0$$

$$(\lambda + \mu) p_{1,n} = \lambda p_{0,n} + (n+1)\gamma p_{0,n+1} + \lambda p_{1,n-1} \qquad n \geq 1$$

$$(\lambda + \mu) p_{1,0} = \lambda p_{0,0} + \gamma p_{0,1}$$

Define $\quad P_0(z) = \sum_{n=0}^{\infty} z^n p_{0,n} \qquad P_1(z) = \sum_{n=0}^{\infty} z^n p_{1,n}$

Then $\quad \lambda P_0(z) + z\gamma P_0'(z) = \mu P_1(z) \qquad$ (A)

and $\quad (\lambda + \mu) P_1(z) = \lambda P_0(z) + \gamma P_0'(z) + \lambda z P_1(z) \qquad$ (B)

Eliminating $P_0'(z)$ in the above, we get $\quad P_1(z) = P_0(z) \dfrac{\rho}{(1-\rho z)} \qquad \rho = \dfrac{\lambda}{\mu}$

Using this in (A), gives $\quad \dfrac{P_0'}{P_0} = \dfrac{\lambda \rho}{\gamma(1-\rho z)}$

This can be solved to get $\quad P_0(z) = C(1-\rho z)^{-\frac{\lambda}{\gamma}} \quad P_1(z) = C\rho(1-\rho z)^{-\frac{\lambda}{\gamma}-1}$ where *C* is an unknown constant of integration. To find *C* we use the fact that $\quad P_0(1) + P_1(1) = 1$.

This gives $\quad C = (1-\rho)^{\left(\frac{\lambda}{\gamma}\right)+1} \qquad P_0(z) = (1-\rho z)\left(\dfrac{1-\rho}{1-\rho z}\right)^{\left(\frac{\lambda}{\gamma}\right)+1} \qquad P_1(z) = \rho\left(\dfrac{1-\rho}{1-\rho z}\right)^{\left(\frac{\lambda}{\gamma}\right)+1}$

The generating function *P*(*z*) of the number of users in orbit will then be given by –

$$P(z) = \sum_{n=0}^{\infty} z^n (p_{0,n} + p_{1,n}) = P_0(z) + P_1(z) = (1+\rho-\rho z)\left(\frac{1-\rho}{1-\rho z}\right)^{\left(\frac{\lambda}{\gamma}\right)+1}$$

(b)

$$\ln P(z) = \left[\frac{\lambda}{\gamma} + 1\right]\left[\ln(1-\rho) - \ln(1-\rho z)\right] + \ln(1+\rho-\rho z)$$

$$\frac{P'(z)}{P(z)} = \left(\frac{\rho\mu}{\gamma} + 1\right)\frac{\rho}{1-\rho z} - \frac{\rho}{1+\rho-\rho z}$$

Using *P*(1)=1 and evaluating the above at z=1, we get

Mean number in orbit = $N_O = P'(1) = \left(\dfrac{\rho\mu}{\gamma} + 1\right)\dfrac{\rho}{1-\rho} - \rho = \dfrac{\rho^2(\mu+\gamma)}{\gamma(1-\rho)}$

(c) **Simple Logic:** If the system is stable then the overall departure from the system must also be at rate λ. This is then also the departure rate from the M/M/1/1 queue and since the queue must be stable (since the system is stable), this must also be the overall rate of arrivals to the M/M/1/1 queue that *actually enter* the queue. The probability of queue occupancy (i.e. one customer in the M/M/1/1 queue) is $\rho = \dfrac{\lambda}{\mu}$.

Mean Number in the M/M/1/1 Queue = ρ

Alternate Method: P{one customer in the queue} = $\sum_{n=1}^{\infty} p_{1n} = P_1(1) = \rho$

Therefore Mean Number in the M/M/1/1 Queue = ρ

2. (a) Solve flow balance to get $\tilde{\lambda} = (2.051\lambda, 2.282\lambda, 1.026\lambda, 0.256\lambda)$

and $\tilde{\rho} = (2.051\rho, 1.141\rho, 1.026\rho, 0.128\rho)$

For the queueing network to be in equilibrium, we require $2.051\rho < 1$

Therefore, $\rho < 0.488$ or $\lambda < 0.488\mu$

(b) For $\lambda = 0.4, \mu = 1$, we get $\tilde{\rho} = (0.82, 0.456, 0.41, 0.051)$

The mean number in each queue $\tilde{N} = (4.568, 0.84, 0.696, 0.054)$

Therefore, the mean total number of jobs in the system is **6.158**

In addition (for the later parts), we would need the mean time spent in each queue. This will be given by $\tilde{W} = (5.568, 0.92, 1.696, 0.527)$

(c) Transit Delay for Jobs Entering at A

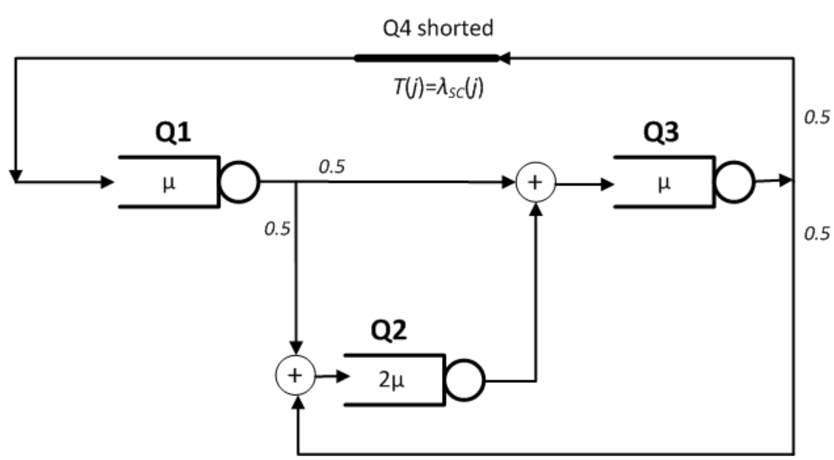
Set the external arrival at B to zero and calculate the flows in each queue. For this, we get

$$\tilde{\lambda} = (0.8204, 0.5128, 0.4104, 0.1024)$$

And $\tilde{V}* = (1.0255, 0.6406, 0.513, 0.128)$

Therefore Transit Delay for a job entering at A will be **7.2368**

3. (a) Since Q4 is the designated sub-network (target queue), we redraw the network with Q4 shorted in order to compute the FES. We then compute *T*(*j*)=λ_{SC}(*j*) as the flow (throughput) through that short for *j*=1,2,….,*M* where *j* is the number of jobs circulating in the network. (Note that *M*=4 in the problem given.)



By flow balance, $\lambda_1 = 0.5\lambda_3 \quad \lambda_2 = 0.5\lambda_1 + 0.5\lambda_3$

Therefore $\lambda_3 = 2\lambda_1$ and $\lambda_2 = 1.5\lambda_1$ $\qquad T(j) = \lambda_{SC}(j) = 0.5\lambda_3 = \lambda_1$

Choosing Q1 as the reference queue with $\lambda_1 = \mu$, we get

Relative Throughputs $\lambda_1 = \mu \quad \lambda_2 = 1.5\mu \quad \lambda_3 = 2\mu$

Visit Ratios $V_1 = 1 \quad V_2 = 1.5 \quad V_3 = 2$

Relative Utilizations $u_1 = 1 \quad u_2 = 0.75 \quad u_3 = 2$

Initialization $N_1 = 0 \quad N_2 = 0 \quad N_3 = 0$

Recursion

$W_1(1) = 1 \quad W_2(1) = 0.5 \quad W_3(1) = 1$

m=1 $\lambda_1^*(1) = 0.267 \quad \lambda_2^*(1) = 0.400 \quad \lambda_3^*(1) = 0.533 \quad \lambda_{SC}(1) = 0.267$

$N_1(1) = 0.267 \quad N_2(1) = 0.200 \quad N_3(1) = 0.533$

$W_1(2) = 1.267 \quad W_2(2) = 0.6 \quad W_3(2) = 1.533$

m=2 $\lambda_1^*(2) = 0.382 \quad \lambda_2^*(2) = 0.573 \quad \lambda_3^*(2) = 0.764 \quad \lambda_{SC}(2) = 0.382$

$N_1(2) = 0.484 \quad N_2(2) = 0.344 \quad N_3(2) = 1.172$

$W_1(3) = 1.484 \quad W_2(3) = 0.672 \quad W_3(3) = 2.172$

m=3 $\lambda_1^*(3) = 0.439 \quad \lambda_2^*(3) = 0.658 \quad \lambda_3^*(3) = 0.878 \quad \lambda_{SC}(3) = 0.439$

$N_1(3) = 0.651 \quad N_2(3) = 0.442 \quad N_3(3) = 1.906$

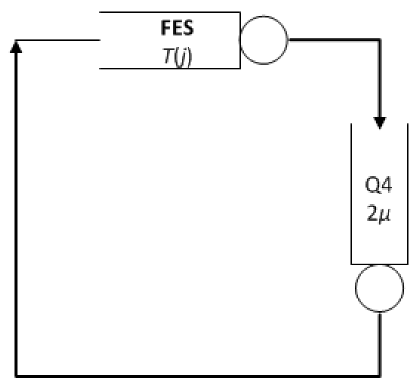
$W_1(4) = 1.651 \quad W_2(4) = 0.721 \quad W_3(4) = 2.906$

m=4 $\lambda_1^*(4) = 0.468 \quad \lambda_2^*(4) = 0.702 \quad \lambda_3^*(4) = 0.936 \quad \lambda_{SC}(4) = 0.468$

$N_1(4) = 0.773 \quad N_2(4) = 0.506 \quad N_3 4) = 2.721$

The State Dependent Service Rates for the corresponding FES would be

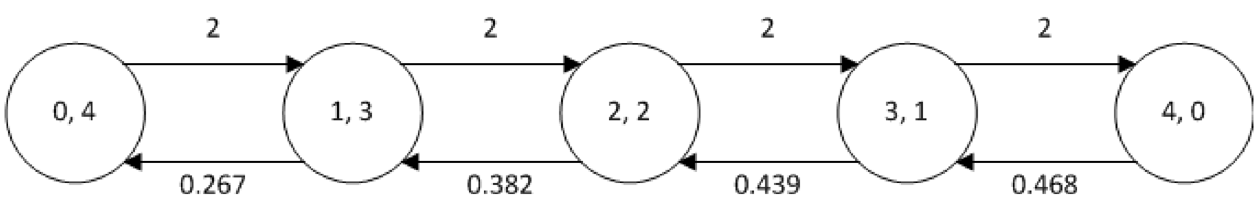
***T*(1)=0.267** ***T*(2)=0.382** ***T*(3)=0.439** ***T*(4)=0.468**



(b) We can define the system state as (N_{FES}, N_{Q4})

where $N_{FES} + N_{Q4} = 4$

The corresponding State Transition Diagram will be as shown below



Solving this –

$$0.267\,p_{1,3} = 2\,p_{0,4} \qquad p_{1,3} = 7.491\,p_{0,4}$$
$$0.382\,p_{2,2} = 2\,p_{1,3} \qquad p_{2,2} = 39.22\,p_{0,4}$$
$$0.439\,p_{3,1} = 2\,p_{2,2} \qquad p_{3,1} = 178.68\,p_{0,4}$$
$$0.468\,p_{4,0} = 2\,p_{3,1} \qquad p_{4,0} = 763.59\,p_{0,4}$$

Therefore

$$P_4 = p_{0,4} = 0.001$$
$$P_3 = p_{1,3} = 0.0076$$
$$P_2 = p_{2,2} = 0.0396$$
$$P_1 = p_{3,1} = 0.1805$$
$$P_0 = p_{4,0} = 0.7713$$

4. M/G/1 queue with vacations AND exceptional first service

P{no arrival in a vacation interval} = $\int_0^\infty e^{-\lambda v} f_V(v)dv = L_V(\lambda)$

Mean Number of Vacations in a Cycle = $\frac{1}{1-L_V(\lambda)}$

Mean Number of Jobs starting the Busy Period in a cycle = $\frac{\lambda\bar{V}}{1-L_V(\lambda)}$

(a) Mean Length of the Busy Period = $(\bar{X}+\Delta)+\lambda(\bar{X}+\Delta)\frac{\bar{X}}{1-\lambda\bar{X}}+\left[\frac{\lambda\bar{V}}{1-L_V(\lambda)}-1\right]\frac{\bar{X}}{1-\lambda\bar{X}}$

$$=\frac{\bar{X}+\Delta}{1-\lambda\bar{X}}+\left[\frac{\lambda\bar{V}+L_V(\lambda)-1}{1-L_V(\lambda)}\right]\frac{\bar{X}}{1-\lambda\bar{X}}$$

$$=\left(\frac{\bar{X}}{1-\lambda\bar{X}}\right)\left[\frac{\lambda\bar{V}}{1-L_V(\lambda)}\right]+\frac{\Delta}{1-\lambda\bar{X}} \quad =\overline{BP}$$

(b) Mean Cycle Length $=\left(\frac{\bar{X}}{1-\lambda\bar{X}}\right)\left[\frac{\lambda\bar{V}}{1-L_V(\lambda)}\right]+\frac{\Delta}{1-\lambda\bar{X}}+\frac{\bar{V}}{1-L_V(\lambda)}$

$$=\frac{\bar{V}}{(1-\lambda\bar{X})(1-L_V(\lambda))}+\frac{\Delta}{1-\lambda\bar{X}} \quad =\overline{T_{cycle}}$$

(c) Note that in a standard M/G/1 queue the busy period length is $\frac{\bar{X}}{1-\lambda\bar{X}}$ and therefore the mean number of customers served in a busy period is $1+\frac{\lambda\bar{X}}{1-\lambda\bar{X}}=\frac{1}{1-\lambda\bar{X}}$. Drawing an analogy from there, the mean number served in this queue will be –

$$\overline{N_{BP}}=1+\frac{\lambda(\bar{X}+\Delta)}{1-\lambda\bar{X}}+\left[\frac{\lambda\bar{V}}{1-L_V(\lambda)}-1\right]\frac{1}{1-\lambda\bar{X}}$$

$$=\frac{1+\lambda\Delta}{1-\lambda\bar{X}}+\left[\frac{\lambda\bar{V}}{1-L_V(\lambda)}-1\right]\frac{1}{1-\lambda\bar{X}}$$

$$=\frac{\lambda\Delta}{1-\lambda\bar{X}}+\left[\frac{\lambda\bar{V}}{1-L_V(\lambda)}\right]\frac{1}{1-\lambda\bar{X}}$$

Alternative (Trivial) Argument : Mean number served in a cycle must be equal to the mean number of arrivals in a cycle for the queue to be in equilibrium. Therefore

$$\overline{N_{BP}}=\lambda\overline{T_{cycle}}=\frac{\lambda\bar{V}}{(1-\lambda\bar{X})(1-L_V(\lambda))}+\frac{\lambda\Delta}{1-\lambda\bar{X}}$$

(d) To find the mean effective service time, we can find the average over a cycle. This gives –

Mean Service Time $\frac{\bar{X}+\Delta+\bar{X}(\overline{N_{BP}}-1)}{\overline{N_{BP}}}=\bar{X}+\frac{\Delta}{\overline{N_{BP}}}$ which is also trivially obvious!