EC633 Queueing Systems
Solutions for the Final Examination

1. The Markov Chain for this may be written as follows.

For $n_i=0$	$n_{i+1}=a_{i+1}$	Probability 0.5
	$= a_{i+1}+1$	Probability 0.5
For $n_i\geq 1$	$n_{i+1}=n_i+ a_{i+1}-1$	Probability 0.5
	$= n_i+ a_{i+1}$	Probability 0.5

(**a**) Directly taking means of the LHS and the RHS of the Markov Chain expressions at equilibrium and using $E(n_i\}=E\{n_{i+1}\}=N$ and $E\{a_{i+1}\}=\lambda\overline{X}=\rho$, we get

$$N = N+\lambda\overline{X}+0.5p_0-0.5(1-p_0)$$
$$0=\rho+p_0-0.5$$

therefore $p_0=0.5-\rho$

(b) Using $A(z)=L_B(\lambda-\lambda z)$, we get

$$P(z)=A(z)p_0[0.5+0.5z]+A(z)0.5z^{-1}[P(z)-p_0]+0.5A(z)[P(z)-p_0]$$
$$zP(z)=0.5zp_0(1+z)A(z)+0.5(1+z)A(z)[P(z)-p_0]$$
$$P(z)=p_0\frac{0.5(1-z)(1+z)A(z)}{[0.5(1+z)A(z)-z]}=p_0\frac{(1-z^2)L_B(\lambda-\lambda z)}{[(1+z)L_B(\lambda-\lambda z)-2z]}$$

Therefore $$P(z)=\frac{(0.5-\rho)(1-z^2)L_B(\lambda-\lambda z)}{(1+z)L_B(\lambda-\lambda z)-2z}$$

2. Let P_B be the probability of blocking and $\rho=\lambda\overline{X}$
Then $p_0=1-\rho(1-P_B)$ and also $p_0=1-P_B$
Therefore, $1-P_B=\frac{1}{1+\rho}$ or $P_B=\frac{\rho}{1+\rho}=\frac{\lambda\overline{X}}{1+\lambda\overline{X}}$

3. In one busy-idle cycle, Mean Length of Idle Period $\overline{I}=\frac{K}{\lambda}=\frac{K\overline{X}}{\rho}$

Let Mean Length of Busy Period = $\overline{BP}$

Then

$$\overline{BP}=\overline{X}+\Delta+\lambda\left(\overline{X}+\Delta\right)\left(\frac{\overline{X}}{1-\rho}\right)+(K-1)\left[\overline{X}+\lambda\overline{X}\left(\frac{\overline{X}}{1-\rho}\right)\right]$$
$$=\frac{K\overline{X}+\Delta}{1-\rho}$$

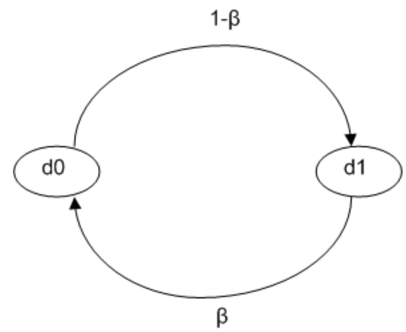
Therefore, P{Server Busy} = $\frac{\overline{BP}}{\overline{I}+\overline{BP}}=\frac{K\rho+\lambda\Delta}{K+\lambda\Delta}$ or $\frac{(K\overline{X}+\Delta)\rho}{(K\overline{X}+\rho\Delta)}$

4. (a) States at the departure instants are $d0$ and $d1$. For notational convenience, let $\rho=\lambda T$ and $\beta=\exp(-\lambda T)$. Let P_B be the probability that an arrival is blocked (because the system is full)

Balance Equation is $p_{do}(1-\beta) = p_{d1}\beta$

Normalization Condition $p_{d0} + p_{d1} = 1$

Therefore, $p_{d0} = \beta = e^{-\lambda T}$ $p_{d1} = 1-\beta = 1-e^{-\lambda T}$



(b) Using Kleinrock's Result,

$$p_{ac0} = p_{d0} = \beta \quad p_{ac1} = p_{d1} = 1-\beta$$

Therefore, from the point of view of an arrival to the system (whether or not it is allowed to enter), we get

$$p_{a0} = (1-P_B)p_{ac0} = (1-P_B)\beta \quad p_{a1} = (1-P_B)p_{ac1} = (1-P_B)(1-\beta) \quad p_{a2} = P_B$$

Using PASTA (since the arrivals come from a Poisson process), $p_i = p_{ai} \quad i = 0,1,2$

Therefore, $p_0 = (1-P_B)\beta \quad p_1 = (1-P_B)(1-\beta) \quad p_2 = P_B$

To find P_B, note that the traffic actually entering the queue is $\lambda_C = \lambda(1-P_B)$

Therefore, $p_0 = 1-\lambda_C T = 1-\rho(1-P_B)$

Equating the expressions for p_0 $\quad 1-\rho(1-P_B) = (1-P_B)\beta$

Therefore $\quad P_B = 1-\dfrac{1}{\rho+\beta} = 1-\dfrac{1}{\lambda T + e^{-\lambda T}}$

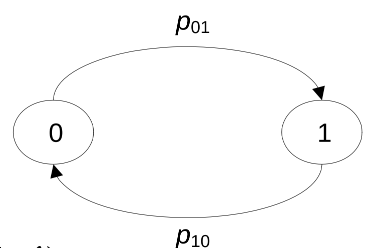
5. For notational convenience, let $\beta = 1-\lambda$

(a) Late Arrival Model: The states at the departure instants are 0 and 1

Transition Probabilities

$$p_{01} = \sum_{k=1}^{\infty} b_k[1-(1-\lambda)^k] = 1-B(\beta)$$

$$p_{10} = \sum_{k=1}^{\infty} b_k(1-\lambda)^k = B(\beta)$$



Balance Equation $\quad p_0[1-B(\beta)] = p_1 B(\beta)$

Normalization $\quad p_0 + p_1 = 1$

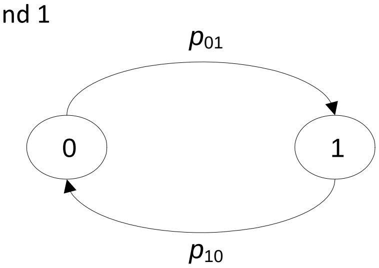
Solving these, we get $\quad p_0 = B(\beta) = B(1-\lambda) \quad p_1 = 1-B(\beta) = 1-B(1-\lambda)$

(b) Early Arrival Model: The states at the departure instants are 0 and 1

Transition Probabilities

$$p_{01} = \sum_{k=1}^{\infty} b_k[1-(1-\lambda)^{k-1}] = 1-\frac{B(\beta)}{\beta}$$

$$p_{10} = \sum_{k=1}^{\infty} b_k(1-\lambda)^k = B(\beta)$$



Balance Equation $\quad p_0[1-\dfrac{B(\beta)}{\beta}] = p_1 B(\beta) \quad p_0 = p_1\dfrac{\beta B(\beta)}{\beta - B(\beta)}$

Normalization $\quad p_0 + p_1 = 1$

Solving these, we get $\quad p_0 = \dfrac{\beta B(\beta)}{\beta B(\beta)+\beta-B(\beta)} \quad p_1 = \dfrac{\beta - B(\beta)}{\beta B(\beta)+\beta-B(\beta)}$

6. (a) Using flow balance conditions in the network, we get -

$\lambda_1 = 2\Lambda + 0.2\lambda_3 + 0.2\lambda_4$	$\lambda_2 = 1.25\Lambda$	$0.8\lambda_1 = 2.2\Lambda$
$\lambda_3 = 0.5\lambda_1$	$0.9\lambda_1 - 0.2\lambda_4 = 2\Lambda$	$\lambda_1 = 2.75\Lambda$
$\lambda_2 = \Lambda + 0.2\lambda_2 \quad \lambda_2 = 1.25\Lambda$	$-0.5\lambda_1 + \lambda_4 = \Lambda$	$\lambda_3 = 1.375\Lambda$
$\lambda_4 = 0.8\lambda_2 + 0.5\lambda_1$		$\lambda_4 = 2.375\Lambda$

Therefore, offered traffic $\lambda_1 = 2.75\Lambda \quad \lambda_2 = 1.25\Lambda \quad \lambda_3 = 1.375\Lambda \quad \lambda_4 = 2.375\Lambda$

and load $\rho_1 = 1.375\frac{\Lambda}{\mu} \quad \rho_2 = 1.25\frac{\Lambda}{\mu} \quad \rho_3 = 1.375\frac{\Lambda}{\mu} \quad \rho_4 = 1.1875\frac{\Lambda}{\mu}$

System will be stable if $1.375\frac{\Lambda}{\mu} < 1$ or $\Lambda < 0.727\mu$

(b) For Λ=0.5, μ=1 $\rho_1 = 0.6875 \quad \rho_2 = 0.625 \quad \rho_3 = 0.6875 \quad \rho_4 = 0.59375$

$N_1 = 2.2 \quad N_2 = 1.667 \quad N_3 = 2.2 \quad N_4 = 1.462$

$N = 7.529$ and $W = \frac{N}{3\Lambda} = \frac{7.529}{3\times 0.5} = 5.019$

(c) For Λ=0.5, μ=1 we get that

$W_1 = \frac{N_1}{\lambda_1} = 1.6 \quad W_2 = \frac{N_2}{\lambda_2} = 2.667 \quad W_3 = \frac{N_3}{\lambda_3} = 3.2 \quad W_4 = \frac{N_4}{\lambda_4} = 1.231$

Considering the system with arrivals coming only from **A**, we get -

$\lambda_1 = 2\Lambda + 0.2\lambda_3 + 0.2\lambda_4 \qquad \lambda_2 = 0 \qquad \lambda_3 = 1.25\Lambda$

$\lambda_3 = 0.5\lambda_1 \qquad 0.9\lambda_1 - 0.2\lambda_4 = 2\Lambda \qquad \lambda_4 = 1.25\Lambda$

$\lambda_2 = 0 \qquad 0.8\lambda_1 = 2\Lambda$

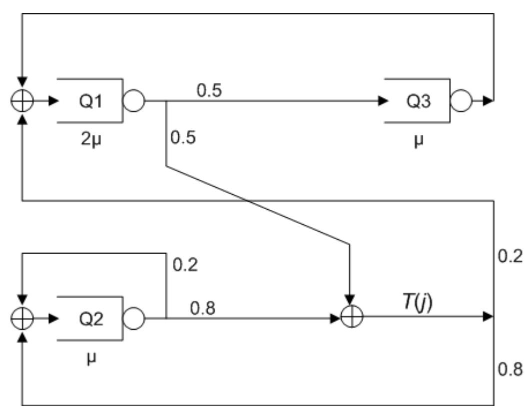
$\lambda_4 = 0.5\lambda_1 \qquad \lambda_1 = 2.5\Lambda$

Therefore, $\lambda_1 = 2.5\Lambda \quad \lambda_2 = 0 \quad \lambda_3 = 1.25\Lambda \quad \lambda_4 = 1.25\Lambda$

Visit Ratios $V_1 = \frac{2.5\Lambda}{2\Lambda} = 1.25 \quad V_2 = 0 \quad V_3 = \frac{1.25\Lambda}{2\Lambda} = 0.625 \quad V_4 = \frac{1.25\Lambda}{2\Lambda} = 0.625$

Therefore, Transit Delay for job entering at A =1.25x1.6+0.625x3.2+0.625x1.231=4.77

7. Since Q4 is the designated sub-network, for computation of the FES, we need to redraw the network with Q4 shorted as shown. We then compute *T*(*j*) as the throughput through that short for *j*=1,2,…,M where *j* is the number of jobs circulating in the network. (Note M=4)



It is best to use MVA for this as that will directly give us the throughput of each queue for *j*=1,2,3,4.

We can then compute $T(j) = 0.5\lambda_1^*(j) + 0.8\lambda_2^*(j)$

$\lambda_1 = \lambda_3 + 0.2(0.5\lambda_1 + 0.8\lambda_2) \qquad 0.9\lambda_1 = 0.5\lambda_1 + 0.16\lambda_2$

$\lambda_2 = 0.2\lambda_2 + 0.8(0.5\lambda_1 + 0.8\lambda_2) \qquad 0.16\lambda_2 = 0.4\lambda_1$

$\lambda_3 = 0.5\lambda_1 \quad \lambda_2 = 2.5\lambda_1$

Choosing Q1 as the reference queue with $\lambda_1 = 2\mu$, we get $\lambda_2 = 5\mu \;\; \lambda_3 = \mu$

Relative Throughputs $\lambda_1 = 2\mu \quad \lambda_2 = 5\mu \quad \lambda_3 = \mu$

Visit Ratios $V_1 = 1 \quad V_2 = 2.5 \quad V_3 = 0.5$

Relative Utilizations $u_1 = 1 \quad u_2 = 5 \quad u_3 = 1$

Initialization $N_1 = 0 \quad N_2 = 0 \quad N_3 = 0$

Recursion

$W_1(1) = 0.5 \quad W_2(1) = 1 \quad W_3(1) = 1$

j=1 $\lambda = \frac{1}{3.5} = 0.2857 \qquad \lambda_1^*(1) = 0.2857 \quad \lambda_2^*(1) = 0.71425 \quad \lambda_3^*(1) = 0.14285$

$N_1(1) = 0.14285 \quad N_2(1) = 0.71425 \quad N_3(1) = 0.14285$

$W_1(2) = 0.5714 \quad W_2(2) = 1.7143 \quad W_3(2) = 1.1429$

j=2 $\lambda = 0.3684 \qquad \lambda_1^*(2) = 0.3684 \quad \lambda_2^*(2) = 0.921 \quad \lambda_3^*(2) = 0.1842$

$N_1(2) = 0.2105 \quad N_2(2) = 1.579 \quad N_3(2) = 0.2105$

$W_1(3) = 0.6053 \quad W_2(3) = 2.579 \quad W_3(3) = 1.2105$

j=3 $\lambda = 0.3918 \qquad \lambda_1^*(3) = 0.3918 \quad \lambda_2^*(3) = 0.9794 \quad \lambda_3^*(3) = 0.1959$

$N_1(3) = 0.2371 \quad N_2(3) = 2.5258 \quad N_3(3) = 0.2371$

$W_1(4) = 0.6186 \quad W_2(4) = 3.5258 \quad W_3(4) = 1.2371$

j=4=M $\lambda = 0.3979 \qquad \lambda_1^*(4) = 0.3979 \quad \lambda_2^*(4) = 0.9949 \quad \lambda_3^*(4) = 0.199$

$N_1(4) = 0.2462 \quad N_2(4) = 3.5077 \quad N_3(4) = 0.2462$

Using $T(j) = 0.5\lambda_1^*(j) + 0.8\lambda_2^*(j)$, we get

$$T(1) = 0.5 * 0.2857 + 0.8 * 0.71425 = 0.71425$$
$$T(2) = 0.5 * 0.3684 + 0.80 * 0.921 = 0.921$$
$$T(3) = 0.5 * 0.3918 + 0.8 * 0.9794 = 0.9794$$
$$T(4) = 0.5 * 0.3979 + 0.8 * 0.9949 = 0.9949$$

as the required FES

(Note that since $\lambda_2^* = 2.5\lambda_1^*$, we can easily see that $T(j) = \lambda_2^*(j)$)