EE633 Queueing Systems (2016-17F)
Solutions to the End-Semester Examination

1.

(a) For Class 2 jobs, the queue will behave like a simple M/G/1/2 queue and the state probabilities can be found accordingly. We then have the following transition probabilities at the departure instants –

$$p_{d,00} = \alpha_0 = L_{B2}(\lambda_2) \qquad p_{d,01} = 1 - L_{B2}(\lambda_2)$$
$$p_{d,10} = L_{B2}(\lambda_2) \qquad p_{d,11} = 1 - L_{B2}(\lambda_2)$$

Balance Equation: $p_{d0} = p_{d0} L_{B2}(\lambda_2) + p_{d1} L_{B2}(\lambda_2) \;\Rightarrow\; p_{d1} = p_{d0} \dfrac{[1 - L_{B2}(\lambda_2)]}{L_{B2}(\lambda_2)}$

Normalization Condition: $p_{d0} + p_{d1} = 1$

Therefore, state probabilities of the Class 2 customers at the departure instants of the Class 2 customers are - $p_{d0} = L_{B2}(\lambda_2) \qquad p_{d1} = 1 - L_{B2}(\lambda_2)$

Traffic actually offered to the queue $\rho_{c2} = \rho_2 (1 - P_{B2})$ where $\rho_2 = \lambda_2 \overline{X_2}$

Therefore, $p_0 = 1 - \rho_2 (1 - P_{B2}) = (1 - P_{B2}) p_{d0} \;\Rightarrow\; 1 - P_{B2} = \dfrac{1}{p_{d0} + \rho_2} = \dfrac{1}{L_{B2}(\lambda_2) + \rho_2}$

Using these,

$$p_0 = (1 - P_{B2}) p_{d0} = \frac{L_{B2}(\lambda_2)}{\rho_2 + L_{B2}(\lambda_2)}$$
$$p_1 = (1 - P_{B2}) p_{d1} = \frac{1 - L_{B2}(\lambda_2)}{\rho_2 + L_{B2}(\lambda_2)} \qquad \textbf{[4]}$$
$$p_2 = P_{B2} = \frac{\rho_2 + L_{B2}(\lambda_2) - 1}{\rho_2 + L_{B2}(\lambda_2)}$$

will be the required state probabilities at an arbitrary time instant

(b) The queue would be stable for Class 1 customers if the following condition holds.

$$\lambda_1 \overline{X_1} + \lambda_2 (1 - P_{B2}) \overline{X_2} < 1 \;\Rightarrow\; \rho_1 < \frac{L_{B2}(\lambda_2)}{L_{B2}(\lambda_2) + \rho_2} \qquad \textbf{[1]}$$

(c) Let α = P{no Class 2 arrivals in a Class 2 service time} = $\int_0^\infty e^{-\lambda_2 x} b_2(x) dx = L_{B2}(\lambda_2)$

Therefore $\overline{BP_2} = \sum_{j=1}^{\infty} j \overline{X_2} (1 - \alpha)^{j-1} \alpha = \dfrac{\overline{X_2}}{\alpha} = \dfrac{\overline{X_2}}{L_{B2}(\lambda_2)}$ **[1]**

$$L_{BP2}(s) = E\{e^{-s(BP2)}\} = \sum_{n=1}^{\infty} L_{B2}^n(s)(1 - \alpha)^{n-1} \alpha = \left(\frac{\alpha}{1 - \alpha}\right)\left(\frac{(1 - \alpha) L_{B2}(s)}{1 - (1 - \alpha) L_{B2}(s)}\right)$$
$$= \frac{L_{B2}(\lambda_2) L_{B2}(s)}{[1 - L_{B2}(s)] + L_{B2}(s) L_{B2}(\lambda_2)} \qquad \textbf{[4]}$$

(d) $$\bar{T} = \overline{X_1} + \lambda_2 \overline{X_1}(\overline{BP2}) = \overline{X_1}\left(1 + \frac{\rho_2}{L_{B2}(\lambda_2)}\right) = \overline{X_1}\left(\frac{L_{B2}(\lambda_2) + \rho_2}{L_{B2}(\lambda_2)}\right)$$ **[2]**

$$L_T(s) = E\{e^{-sT}\} = \sum_{n=0}^{\infty} E\left\{e^{-sX_1} L_{BP2}^n(s) \frac{(\lambda_2 X_1)^n}{n!} e^{-\lambda_2 X_1}\right\}$$

$$= E\left\{\sum_{n=0}^{\infty} e^{-(s+\lambda_2)X_1} \frac{[\lambda_2 X_1 L_{BP2}(s)]^n}{n!}\right\}$$ **[4]**

$$= E\left\{e^{-[s+\lambda_2-\lambda_2 L_{BP2}(s)]X_1}\right\}$$

$$= L_{B1}(s + \lambda_2 - \lambda_2 L_{BP2}(s))$$

Note that you can also obtain $\bar{T}$ from $L_T(s)$

(e) In order to use the standard M/G/1 result, note that $\rho = \lambda_1 \overline{T_1} = \rho_1\left(\frac{L_{B2}(\lambda_2) + \rho_2}{L_{B2}(\lambda_2)}\right)$ and

$$A(z) = L_T(\lambda_1 - \lambda_1 z) = L_{B1}(\lambda_1 - \lambda_1 z + \lambda_2 - \lambda_2 L_{BP2}(\lambda_1 - \lambda_1 z))$$

Therefore,

$$P_1(z) = \frac{(1-\rho)(1-z)L_{B1}(\lambda_1 - \lambda_1 z + \lambda_2 - \lambda_2 L_{BP2}(\lambda_1 - \lambda_1 z))}{L_{B1}(\lambda_1 - \lambda_1 z + \lambda_2 - \lambda_2 L_{BP2}(\lambda_1 - \lambda_1 z)) - z}$$ **[4]**

2. The flow balance equations for this system may be written as –

$$\lambda_1 = 2\Lambda + 0.2\lambda_3 + 0.5\lambda_2$$
$$\lambda_2 = \Lambda + 0.2\lambda_4$$
$$\lambda_3 = 0.5\lambda_1$$
$$\lambda_4 = 0.5\lambda_1 + 0.5\lambda_2$$

Solving these, we get $\lambda_1 = 3.03\Lambda, \lambda_2 = 1.45\Lambda, \lambda_3 = 1.52\Lambda, \lambda_4 = 2.24\Lambda$

(a) Traffic Vector for the queues $\vec{\rho} = \left(3.03\frac{\Lambda}{\mu}, 0.725\frac{\Lambda}{\mu}, 0.76\frac{\Lambda}{\mu}, 2.24\frac{\Lambda}{\mu}\right)$

Therefore, the system will be stable if $3.03\frac{\Lambda}{\mu} < 1$ or $\frac{\Lambda}{\mu} < 0.33$ **[2]**

(b) For Λ = 0.3 and μ= 1

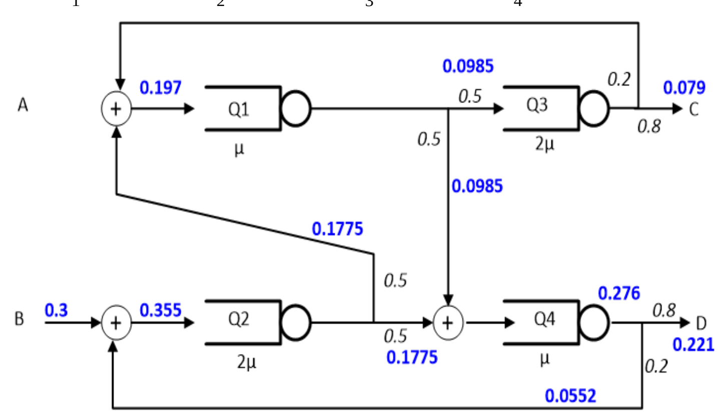
$\lambda_1 = 0.908, \lambda_2 = 0.434, \lambda_3 = 0.455, \lambda_4 = 0.672$ & $\rho_1 = 0.908, \rho_2 = 0.217, \rho_3 = 0.228, \rho_4 = 0.672$

(i) The mean transit delays through the queues may be found by using the corresponding M/M/1 expression. These are $W_1 = 10.87 \quad W_2 = 0.639 \quad W_3 = 0.648 \quad W_4 = 3.049$

The visit ratios to the queues are $V_1 = 1.009 \quad V_2 = 0.482 \quad V_3 = 0.506 \quad V_4 = 0.747$

Therefore, the mean transit time through the system will be ***$W_{AB.CD}$* = 13.88** **[2]**

(ii) To find the mean transit delay for jobs entering the system at **B**, set the flow entering from **A** to zero. The network will then be as shown in the figure.



Then, $\lambda_1 = 0.5\lambda_2 + 0.2\lambda_3,\ \lambda_2 = 0.3 + 0.2\lambda_4,\ \lambda_3 = 0.5\lambda_1,\ \lambda_4 = 0.5\lambda_1 + 0.5\lambda_2$

Therefore, $\lambda_1 = 0.197,\quad \lambda_2 = 0.355,\quad \lambda_3 = 0.099,\quad \lambda_4 = 0.276$

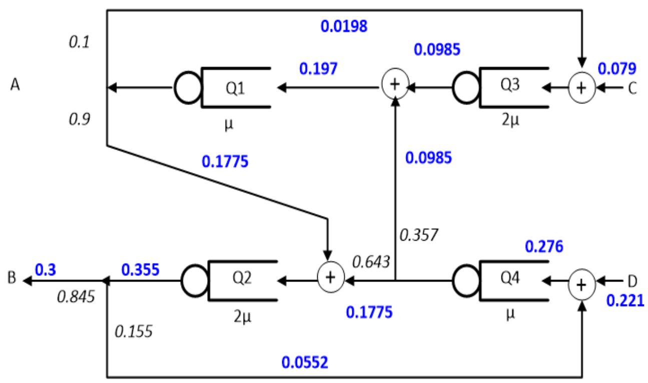
Visit Ratios $V_1 = 0.657,\quad V_2 = 1.183,\quad V_3 = 0.33,\quad V_4 = 0.92$

Using these with the **queue delays calculated in (i)** gives $\boldsymbol{W_{B,CD}}$**=10.916** **[3+3]**

To find the mean delay for jobs entering from **A**, we can repeat the same strategy setting the flow entering at **B** to be zero in the original network. However, it would be much easier to use the following.

$$\frac{0.6}{0.9}W_{A,CD} + \frac{0.3}{0.9}W_{B,CD} = W_{AB,CD} = 13.88 \quad \Rightarrow \quad W_{A,CD} = 15.36$$

(iii) To do this, we first consider the network where jobs enter only from **B** and leave from either **C** or **D**, i.e. the same network as given above in the solution for (ii). We then reverse the network where the flows enter from **C** or **D** and leave through **B**. The reversed network will be as shown.



In this network, we now set the flow entering from **D** to be zero and consider only the flow entering from **C**. (Note that this flow will only leave through **B**.) The flow equations then are –

$$\lambda_1 = \lambda_3 + 0.357\lambda_4 \quad \lambda_2 = 0.9\lambda_1 + 0.643\lambda_4 \quad \lambda_3 = 0.079 + 0.1\lambda_1 \quad \lambda_4 = 0.155\lambda_2$$

Solving these, we get $\lambda_1 = 0.093 \quad \lambda_2 = 0.093 \quad \lambda_3 = 0.088 \quad \lambda_4 = 0.014$

with Visit Ratios $V_1 = 1.177 \quad V_2 = 1.177 \quad V_3 = 1.114 \quad V_4 = 0.177$

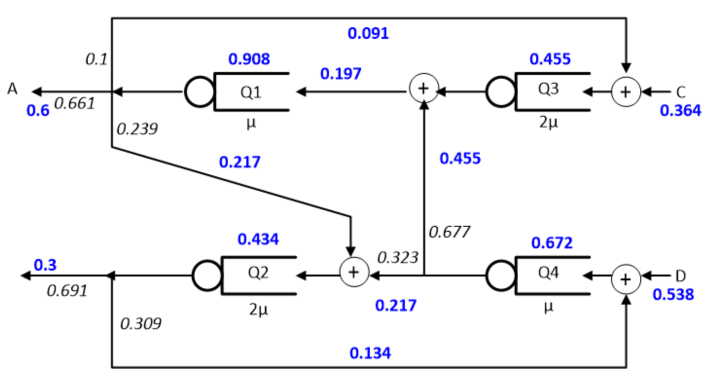
Therefore, the mean transit time $\boldsymbol{W_{B,C}}$**= 14.16**

Note that $\dfrac{0.079}{0.3}W_{B,C} + \dfrac{0.221}{0.3}W_{B,D} = W_{B,CD} = 10.916 \quad \Rightarrow \quad W_{B,D} = 9.756$ **[3+3]**

(Direct analysis gives $W_{B,D}$=9.5. The difference appears to be because of round-off errors.)

(c) Reversing the original network, will give the network shown. **[4]**

Note: The numbers in blue give the actual flow values. The numbers in *italics* (black) give the corresponding routing probabilities



3. (a) Late Arrival Model

Considering the Imbedded Markov Chain at the customer departure instants with the usual notation–

$$n_{i+1} = a_{i+1} \qquad n_i = 0$$
$$\quad = n_i + a_{i+1} - 1 \qquad n_i \ge 1$$

where

$$\bar{n} = p_0\bar{a} + \bar{n} + (1-p_0)\bar{a} - (1-p_0)$$
$$p_0 = 1 - \bar{a} = 1 - \lambda\bar{b}$$

and $P(z) = p_0 A(z) + \frac{A(z)}{z}\left[P(z) - p_0\right] \quad \Rightarrow \quad P(z) = \frac{p_0(1-z)A(z)}{A(z)-z}$

with $A(z) = \sum_{j=1}^{\infty} b(j) \sum_{k=0}^{j} \binom{j}{k} \lambda^k (1-\lambda)^{j-k} z^k = \sum_{j=1}^{\infty} b(j)\left(1-\lambda+\lambda z\right)^k = B(1-\lambda+\lambda z)$ **[6]**

Differentiating and evaluating at z=1, we get-

$$A'(1) = \lambda B'(1) = \lambda \bar{b} \ or \ \lambda b^{(1)} \qquad A''"(1) = \lambda^2 B''(1) = \lambda^2\left(b^{(2)} - \bar{b}\right)$$

Differentiating twice to calculate *N*,

$$(A-z)P = p_0(1-z)A$$

$$P'(A-z) + P(A'-1) = p_0(1-z)A' - p_0 A$$

$$P''(A-z) + 2P'(A'-1) + PA'' = p_0(1-z)A'' - 2p_0 A'$$

Evaluating the last expression for *z*=1,

$$2N(\lambda\bar{b}-1) + \lambda^2\left(b^{(2)} - \bar{b}\right) = -2p_0\lambda\bar{b} = -2(1-\lambda\bar{b})\lambda\bar{b}$$

$$N = \frac{\lambda^2\left(b^{(2)} - \bar{b}\right)}{2(1-\lambda\bar{b})} + \lambda\bar{b}$$ **[6]**

(b) FCFS Queue

For the **Late Arrival** model, the number in the system as seen by a departing user would be the number arriving while that user was in the system. Let $g_W(k) \quad k = 1, \ldots, \infty$ be the probability that the user spent *k* time slots in the system with $G_W(z) = \sum_{k=1}^{\infty} g_W(k) z^k$ as its generating function. Therefore,

$$P(z) = \sum_{k=1}^{\infty} g_W(k) \sum_{j=0}^{k} \binom{k}{j} \lambda^j (1-\lambda)^{k-j} z^j = \sum_{k=1}^{\infty} g_W(k)(1-\lambda+\lambda z)^k = G_W(1-\lambda+\lambda z)$$

Note that in the **Early Arrival** model, a customer spending the same amount of time in the system (as for the Late Arrival model) will count arrivals in one less slot on departure. Therefore,

$$\tilde{P}(z) = \sum_{k=1}^{\infty} g_W(k) \sum_{j=0}^{k-1} \binom{k-1}{j} \lambda^j (1-\lambda)^{k-1-j} z^j = \sum_{k=1}^{\infty} g_W(k)(1-\lambda+\lambda z)^{k-1} = \frac{G_W(1-\lambda+\lambda z)}{(1-\lambda+\lambda z)}$$

Therefore,

$$\tilde{P}(z) = \frac{P(z)}{(1-\lambda+\lambda z)}$$ **[8]**