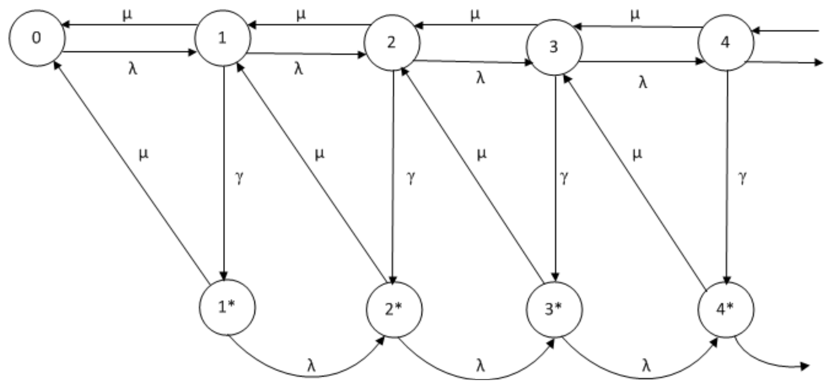
EE633 Queueing Systems (2015-16F)
Solutions to the End-Sem Examination

1.

(a) State Transition Diagram



(b)

$$\mu[p_1 + p_1^*] = \lambda p_0 \qquad (\lambda+\mu)p_1^* = p_1\gamma$$
$$\mu[p_2 + p_2^*] = \lambda p_1 + \lambda p_1^* \qquad (\lambda+\mu)p_2^* = p_2\gamma + p_1^*\lambda$$
$$\mu[p_3 + p_3^*] = \lambda p_2 + \lambda p_2^* \qquad (\lambda+\mu)p_3^* = p_3\gamma + p_2^*\lambda$$
$$\mu[p_4 + p_4^*] = \lambda p_3 + \lambda p_3^* \qquad (\lambda+\mu)p_4^* = p_4\gamma + p_3^*\lambda$$

....

....

(c) Multiplying the n^{th} equation in each set by z^n, n=1,2,3,….. and summing, we get -

$$P(z) + P^*(z) = \rho p_0 z + \rho z[P(z) + P^*(z)]$$ *From the left-hand set*

$$P(z) + P^*(z) = p_0 \frac{\rho z}{1-\rho z}$$ **[A]**

$$(1+\rho)P^*(z) = \alpha P(z) + \rho z P^*(z)$$ *From the right-hand set*

$$P^*(z) = \frac{\alpha}{1+\rho-\rho z} P(z)$$ **[B]**

Equations [A] and [B] can be solved to obtain $P(z)$ and $P^*(z)$ in terms of p_0, ρ and α

$$P(z) = p_0 \frac{\rho z(1+\rho-\rho z)}{(1-\rho z)(1+\rho+\alpha-\rho z)} \quad \text{and} \quad P^*(z) = p_0 \frac{\alpha\rho z}{(1-\rho z)(1+\rho+\alpha-\rho z)}$$

(d) Putting z=1, we get from [A] that $P(1) + P^*(1) = p_0 \frac{\rho}{1-\rho}$

Moreover, the Normalization Condition gives $P(1) + P^*(1) + p_0 = 1$

Therefore, $p_0 = 1-\rho$

[*Can also find this by evaluating the expressions for* $P(z)$ *and* $P^*(z)$ *at z*=1]

(e) Define $\tilde{i}$ as the Generating Function for finding n customers in the system. Then it is evident

that $\tilde{i}$ $+ P(z) + P^*(z) = p_0\left(1 + \frac{\rho z}{1-\rho z}\right) = \frac{1-\rho}{1-\rho z} = \sum_{n=0}^{\infty}(1-\rho)\rho^n z^n$

Therefore, P {n customers in the system} = $\rho^n(1-\rho)$ for n=0, 1, 2 …………

(f) P {Manager is working} = $P^*(1)$ and P{Teller is working}= $P(1)$

Using [A] and [B] evaluated at *z*=1 and p_0 from (d) above, we get that

P {Manager is working} = $P^*(1) = \rho \dfrac{\alpha}{1+\alpha}$

P {Teller is working} = $P(1) = \rho \dfrac{1}{1+\alpha}$

(g) Note that service to a customer always starts with the teller but may end either with the teller or with the manager

$\bar{n} = \sum_{n=1}^{\infty} np_n$ is the mean number of customers whose service ends with the teller

$\overline{n^*} = \sum_{n=1}^{\infty} np_n^*$ is the mean number of customers whose service ends with the manager

$\bar{n} + \overline{n^*}$ is the mean number of customers in the system

2. The flow balance equations for this system may be written as -

$$\begin{aligned} 2\Lambda + 0.2\lambda_3 &= \lambda_1 \\ \Lambda + 0.4\lambda_4 &= \lambda_2 \\ \lambda_4 &= \lambda_2 \\ \lambda_3 &= \lambda_1 + 0.2\lambda_4 \end{aligned}$$

Solving these, we get $\lambda_1 = 2.584\Lambda, \lambda_2 = 1.667\Lambda, \lambda_3 = 2.917\Lambda, \lambda_4 = 1.667\Lambda$

(a) Traffic vector for the queues $\vec{\rho} = (2.584\frac{\Lambda}{\mu}, 0.833\frac{\Lambda}{\mu}, 1.459\frac{\Lambda}{\mu}, 1.667\frac{\Lambda}{\mu})$

Therefore, the system will be stable if $2.584\frac{\Lambda}{\mu} < 1$ or $\frac{\Lambda}{\mu} < 0.387$

(b) For Λ = 0.3 and μ= 1

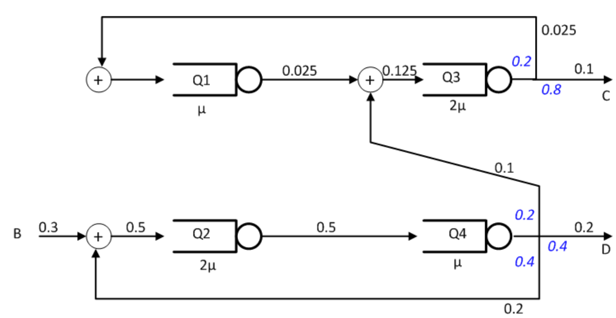
$\lambda_1 = 0.775 \quad \lambda_2 = 0.5 \quad \lambda_3 = 0.875 \quad \lambda_4 = 0.5$ & $\rho_1 = 0.775 \quad \rho_2 = 0.25 \quad \rho_3 = 0.438 \quad \rho_4 = 0.5$

(i) The mean transit delays through each of the queues may be found using the corresponding M/M/1 expressions. These will be $W_1 = 4.444, \quad W_2 = 0.667, \quad W_3 = 0.889, \quad W_4 = 2$

The visit ratios to the queues are $V_1 = 0.861, V_2 = 0.556, V_3 = 0.972, V_4 = 0.556$

Therefore, the mean transit time through the system will be $\mathbf{W_{AB,CD}}$=6.173

(ii) To find the mean transit delay for jobs entering the system from **B**, set the flow entering from **A** to zero. The network will then be as shown in the figure



Then $\lambda_2 = 0.3 + 0.4\lambda_4 \quad \lambda_4 = \lambda_2 \quad \lambda_1 = 0.2\lambda_3 \quad \lambda_3 = \lambda_1 + 0.2\lambda_4$

Therefore, $\lambda_1 = 0.025 \quad \lambda_2 = 0.5 \quad \lambda_3 = 0.125 \quad \lambda_4 = 0.5$

The corresponding visit ratios are $V_1 = 0.083 \quad V_2 = 1.667, \quad V_3 = 0.417, \quad V_4 = 1.667$.

Using these with the individual queue delays calculated in (i) gives the mean transit time for jobs entering from **B** to be **$W_{B,CD}$= 5.185**

One can of course repeat the same strategy by now setting the flow entering from **B** to zero in the original network. However, it is much easier to use the following –

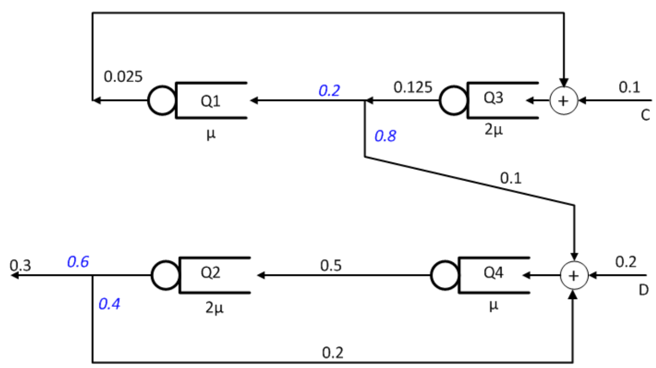
$$\frac{0.6}{0.9} W_{A,CD} + \frac{0.3}{0.9} W_{B,CD} = W_{AB,CD} = 6.173$$

Therefore, **$W_{A,CD}$=6.667**

(iii) To do this, we need to first consider the network where jobs enter only from **B** and leave from **C** or **D**, i.e. with the flow from **A** set to 0. This would be the same network as in (ii)

We then reverse this network so that flows enter from either **C** or **D** and leave through **B**

This reversed network will be as shown.



In this network, we now set the flow entering from **D** to zero and consider only the flow entering from **C.** The flow equations then are –

$\lambda_1 = 0.2\lambda_3 \quad \lambda_3 = 0.1 + \lambda_1$

$\lambda_2 = \lambda_4 \qquad \lambda_4 = 0.4\lambda_2 + 0.8\lambda_3$

$\lambda_1 = 0.025 \quad \lambda_2 = 0.167$

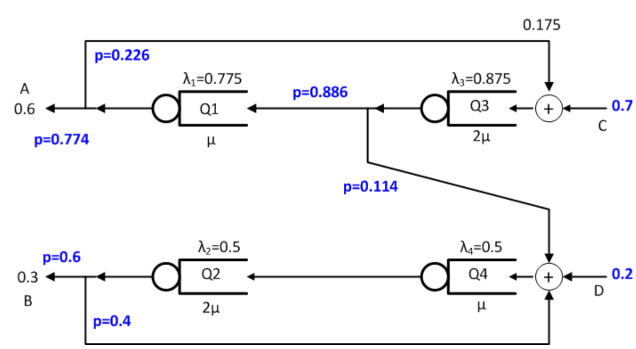
$\lambda_3 = 0.125 \quad \lambda_4 = 0.167$

The corresponding visit ratios are $V_1 = 0.25 \quad V_2 = 1.67 \quad V_3 = 1.25 \quad V_4 = 1.67$ and the corresponding mean transit time **$W_{B,C}$=6.676**

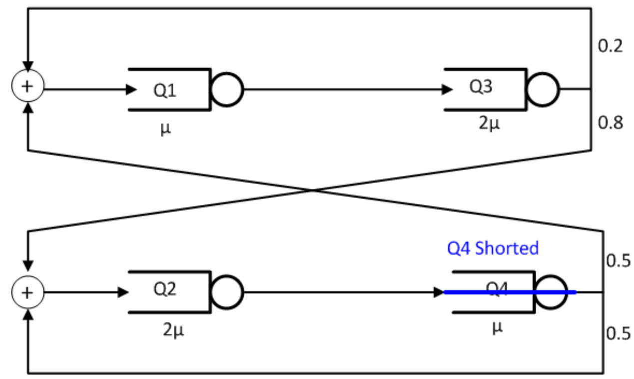
Note that $\frac{0.1}{0.3} W_{B,C} + \frac{0.2}{0.3} W_{B,D} = W_{B,CD} = 5.185 \Rightarrow$ **$W_{B,D}$=4.44**

(*Of course, one can also find this by repeating the strategy to find* $W_{B,C}$)

(c) Reversing the original network gives the network shown



3. We short Q4 since that is the designated (sub) network (i.e. the target queue) and redraw the network as shown to calculate the FES. We then compute $T(j)$=$\lambda_{SC}(j)$ as the flow through that short for j=1,2,3,4 where j is the number of jobs circulating in the network.



(a) By flow balance, we get $\lambda_1 = 0.5\lambda_2 + 0.2\lambda_3 \quad \lambda_3 = \lambda_1$

Therefore,

$\lambda_2 = 1.6\lambda_1 \quad \lambda_3 = \lambda_1 \quad T(j) = \lambda_{SC}(j) = \lambda_2 = 1.6\lambda_1$

Choosing Q1 as the reference queue with $\lambda_1 = \mu$,

Relative Throughputs $\lambda_1 = \mu \quad \lambda_2 = 1.6\mu \quad \lambda_3 = \mu$

Visit Ratios $V_1 = 1 \quad V_2 = 1.6 \quad V_3 = 1$

Relative Utilizations $u_1 = 1 \quad u_2 = 0.8 \quad u_3 = 0.5$

Initialization $N_1 = 0 \quad N_2 = 0 \quad N_3 = 0$

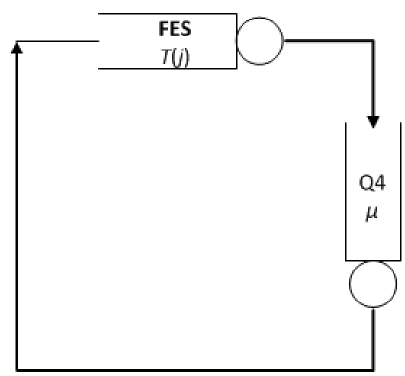
Recursion

m=1

$$W_1 = 1 \quad W_2 = 0.5 \quad W_3 = 0.5$$
$$\lambda_1^*(1) = 0.435 \quad \lambda_2^*(1) = 0.696 \quad \lambda_3^*(1) = 0.435$$
$$N_1(1) = 0.435 \quad N_2(1) = 0.348 \quad N_3(1) = 0.217$$

m=2

$$W_1 = 1.435 \quad W_2 = 0.674 \quad W_3 = 0.609$$
$$\lambda_1^*(2) = 0.641 \quad \lambda_2^*(2) = 1.025 \quad \lambda_3^*(2) = 0.641$$
$$N_1(2) = 0.919 \quad N_2(2) = 0.691 \quad N_3(2) = 0.390$$



m=3

$$W_1 = 1.919 \quad W_2 = 0.845 \quad W_3 = 0.695$$
$$\lambda_1^*(3) = 0.756 \quad \lambda_2^*(3) = 1.210 \quad \lambda_3^*(3) = 0.756$$
$$N_1(3) = 1.451 \quad N_2(3) = 1.023 \quad N_3(3) = 0.526$$

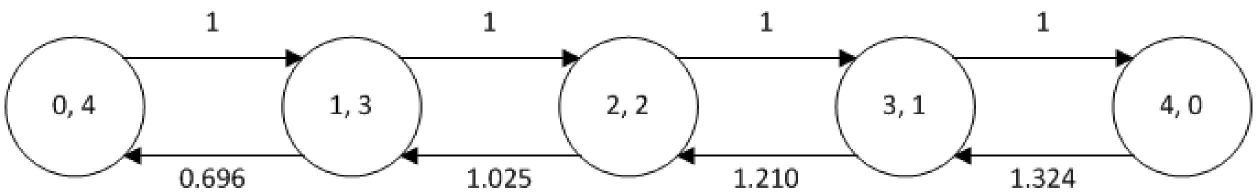
m=4

$$W_1 = 2.451 \quad W_2 = 1.011 \quad W_3 = 0.763$$
$$\lambda_1^*(4) = 0.828 \quad \lambda_2^*(4) = 1.324 \quad \lambda_3^*(4) = 0.828$$
$$N_1(4) = 2.029 \quad N_2(4) = 1.340 \quad N_3(4) = 0.631$$

The state dependent service rates for the corresponding FES will be

$$T(1) = 0.696 \quad T(2) = 1.025 \quad T(3) = 1.210 \quad T(4) = 1.324$$

(b) We define the system state as (N_{FES}, N_{Q4}) where N_{FES}+N_{Q4}=4. The corresponding state transition diagram is given below.



Solving this -

$$
\begin{aligned}
0.696\,p_{1,3} &= p_{0,4} & p_{1,3} &= 1.437\,p_{0,4} \\
1.025\,p_{2,2} &= p_{1,3} & p_{2,2} &= 1.402\,p_{0,4} \\
1.210\,p_{3,1} &= p_{2,2} & p_{3,1} &= 1.158\,p_{0,4} \\
1.324\,p_{4,0} &= p_{3,1} & p_{4,0} &= 0.875\,p_{0,4}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P_4 &= p_{0,4} = 0.1703 \\
P_3 &= p_{1,3} = 0.2447 \\
P_2 &= p_{2,2} = 0.2388 \\
P_1 &= p_{3,1} = 0.1972 \\
P_0 &= p_{4,0} = 0.1490
\end{aligned}
$$