EE 633, Queueing Systems
Final Examination (2012-2013S)
Solutions

1. (a) Let the *Effective Service Time Distribution* (Laplace Transform) of a job entering the system (from outside) be $L_{B^*}(s)$.

Therefore $L_{B^*}(s)=\sum_{k=1}^{\infty} pL_B(s)\left[qL_B(s)\right]^{k-1}=\frac{pL_B(s)}{1-qL_B(s)}$

with Mean Effective Service Time $\overline{X^*}=\sum_{k=1}^{\infty} pq^{k-1}(k\overline{X})=\frac{p\overline{X}}{(1-q)^2}=\frac{\overline{X}}{p}$

where $\overline{X}$ is the mean service time in the actual queue.

Effective Traffic = $\rho^*=\lambda\overline{X^*}=\frac{\lambda\overline{X}}{p}=\frac{\rho}{p}$ where $\rho=\lambda\overline{X}$

Following the analysis of an M/G/1 queue (and considering the whole system as a M/G/1 queue with the effective service time distribution $L_{B^*}(s)$ as given earlier), we get –

$$P_B(z)=p_0\frac{(1-z)L_{B^*}(\lambda-\lambda z)}{L_{B^*}(\lambda-\lambda z)-z} \qquad \text{with } p_0=1-\rho^*$$

Simplifying this gives,

$$P_B(z)=(p-\rho)\frac{(1-z)L_B(\lambda-\lambda z)}{(p+qz)L_B(\lambda-\lambda z)-z}$$ **[5]**

(b) For point A, the Markov chain may be written at the departure instants with n_i as the number seen left behind in the system by the i^{th} departure.

For n_i=0	$n_{i+1}=a_{i+1}$	probability = p
	$=a_{i+1}+1$	probability = q
For $n_i\geq 1$	$n_{i+1}=n_i+a_{i+1}-1$	probability = p
	$=n_i+a_{i+1}$	probability = q

Therefore, (with $A(z)$=E[z^a] , note that $A(z)=L_B(\lambda-\lambda z)$)

$$P_A(z)=A(z)p_{A0}[p+qz]+A(z)pz^{-1}[P_A(z)-p_{A0}]+A(z)q[P_A(z)-p_{A0}]$$
$$P_A(z)=p_{A0}\frac{(1-z)(p+qz)A(z)}{[(p+qz)A(z)-z]}$$

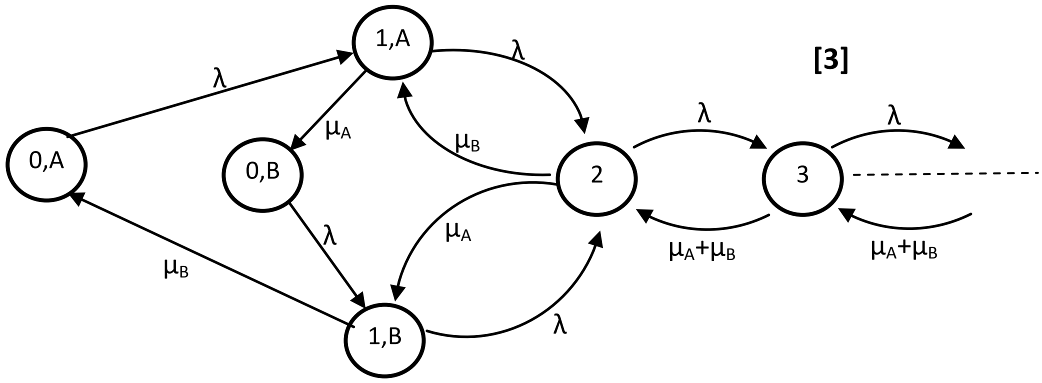
Directly taking means of the LHS and RHS of the Markov Chain expressions at equilibrium and using $E\{n_i\}=E\{n_{i+1}\}=N$ and $E\{a_{i+1}\}=\lambda\overline{X}=\rho$, we get

$$N=N+\lambda\overline{X}+p_{A0}q-(1-p_{A0})p$$
$$0=\rho+p_{A0}-p \qquad \Rightarrow \quad p_{A0}=p-\rho$$

Therefore $P_A(z)=(p-\rho)\frac{(1-z)(p+qz)A(z)}{[(p+qz)A(z)-z]}=(p-\rho)\frac{(1-z)(p+qz)L_B(\lambda-\lambda z)}{[(p+qz)L_B(\lambda-\lambda z)-z]}$ **[5]**

2. A State Transition Diagram for this system is given below. Note that states 2, 3,...∞ are normally defined. The other states are defined as follows –

{1,A} one customer in the system, Server A working
{1,B} one customer in the system, Server B working
{0,A} system empty, Server A idle for longer time than Server B
{0,B} system empty, Server B idle for longer time than Server A



[3]

Solving the balance equations for this we get –

$$p_{0A} = p_{0B} = \frac{\mu_A \mu_B}{\lambda^2} p_2 \qquad p_{1A} = \frac{\mu_B}{\lambda} p_2 \quad p_{1B} = \frac{\mu_A}{\lambda} p_2$$

$$p_n = \left(\frac{\lambda}{\mu_A + \mu_B} \right)^{n-2} p_2 \qquad n = 2,3,4,.......$$

[5]

The normalization condition will then give

$$p_2 = \frac{1}{\left(2\frac{\mu_A \mu_B}{\lambda^2} + \frac{\mu_A + \mu_B}{\lambda} + \frac{\mu_A + \mu_B}{\mu_A + \mu_B - \lambda} \right)}$$

Using p_2 from above, the required probabilities in terms of p_2 are –

$$p_0 = 2\frac{\mu_A \mu_B}{\lambda^2} p_2$$

$$p_1 = \frac{\mu_A + \mu_B}{\lambda} p_2$$

[2]

$$p_n = \left(\frac{\lambda}{\mu_A + \mu_B} \right)^{n-2} p_2 \qquad n = 2,3,4,.......$$

3. (a) Let P_B be the probability of blocking and $\rho = \lambda \overline{X}$

Then $p_0 = 1 - \rho(1 - P_B)$ and also $p_0 = 1 - P_B$

Therefore, $1 - P_B = \dfrac{1}{1+\rho}$

or $P_B = \dfrac{\rho}{1+\rho} = \dfrac{\lambda \overline{X}}{1 + \lambda \overline{X}}$ **[2]**

(b) **(i)** In one busy-idle cycle, Mean Length of Idle Period $\bar{I} = \frac{K}{\lambda} = \frac{K\overline{X}}{\rho}$

Let Mean Length of Busy Period = $\overline{BP}$

Then

$$\overline{BP} = \bar{X} + \Delta + \lambda\left(\bar{X} + \Delta\right)\left(\frac{\bar{X}}{1-\rho}\right) + (K-1)\left[\bar{X} + \lambda\bar{X}\left(\frac{\bar{X}}{1-\rho}\right)\right]$$

$$= \frac{K\bar{X} + \Delta}{1-\rho}$$

and $\bar{T}_{cycle} = \bar{I} + \overline{BP} = \frac{K\overline{X}}{\rho} + \frac{K\bar{X} + \Delta}{1-\rho} = \frac{K\bar{X} + \rho\Delta}{\rho(1-\rho)}$

Therefore P{Server Busy} = $\frac{\overline{BP}}{\bar{I} + \overline{BP}} = \frac{K\rho + \lambda\Delta}{K + \lambda\Delta}$ or $\frac{(K\overline{X} + \Delta)\rho}{(K\overline{X} + \rho\Delta)}$ **[5]**

(ii) P{system empty} = $\frac{\frac{1}{\lambda}}{\bar{I} + \overline{BP}} = \frac{\bar{X}(1-\rho)}{K\bar{X} + \rho\Delta}$ **[3]**

4. Utilizing the symmetry of the network, it is evident that $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$

Also $\lambda_1 = \lambda + 0.5\lambda_2$ and $\lambda_1 = \lambda_2$

Therefore $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2\lambda$

$\tilde{\lambda} = (2\lambda, 2\lambda, 2\lambda, 2\lambda)$ $\tilde{\rho} = (2\rho, \rho, \rho, 2\rho)$ $\tilde{V} = (1,1,1,1)$

(a) For system to be stable, we need 2ρ<1 or ***ρ*<0.5 or *λ*<0.5*μ*** **[2]**

(b) $\tilde{\lambda} = (0.2, 0.2, 0.2, 0.2)$ $\tilde{\rho} = (0.2, 0.1, 0.1, 0.2)$

$\tilde{N} = (0.25, 0.1111, 0.1111, 0.25)$

$\tilde{W} = (1.25, 0.5556, 0.5556, 1.25)$

Therefore **Mean Transit Time = 3.6111** **[3]**

(c) Set λ_B=0 **(**Note that the symmetry used earlier can no longer be used)

$\lambda_3 = 0.5\lambda_4$ $\lambda_4 = 0.5\lambda_3 + 0.5\lambda_1$ $\Rightarrow$ $\lambda_3 = \frac{1}{3}\lambda_1$ $\lambda_4 = \frac{2}{3}\lambda_1$

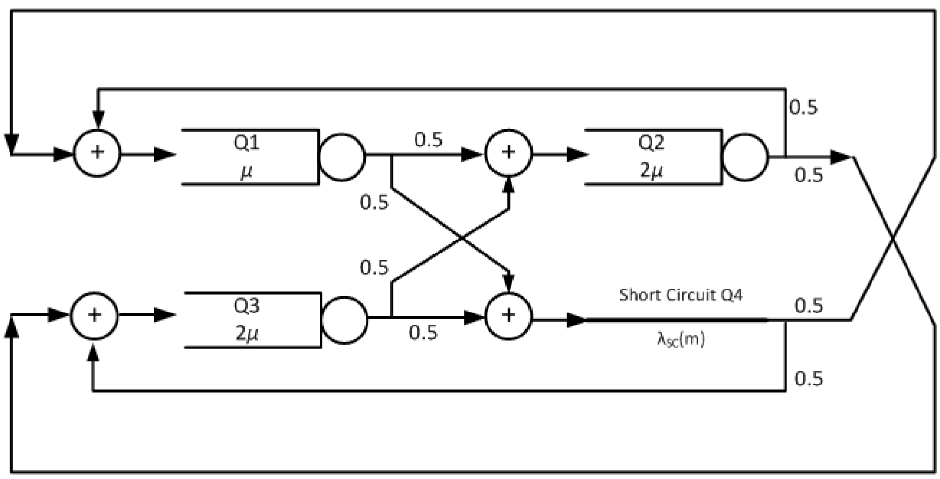
$\lambda_2 = 0.5\lambda_3 + 0.5\lambda_1 = \frac{2}{3}\lambda_1$

$\lambda_1 = \lambda + 0.5\lambda_2$ $\lambda_1 = \lambda + \frac{1}{3}\lambda_1$ $\lambda_1 = 1.5\lambda = 0.15$

$\tilde{\lambda} = (0.15, 0.1, 0.05, 0.1)$ $\tilde{V} = (1.5, 1, 0.5, 1)$

Mean Transit Time for jobs entering from A =3.9584 **[5]**

5. (a) Since Q4 is the designated sub-network, for computation of the FES, we need to redraw the network with Q4 shorted. We then compute $T(j)$ as the throughput through that short for j=1,2,...,M where j is the number of jobs circulating in the network. (Note M=4)



Choose Q1 as the reference queue

$$\lambda_1 = 0.5\lambda_2 + 0.5[0.5\lambda_1 + 0.5\lambda_3] \qquad 0.75\lambda_1 = 0.5\lambda_2 + 0.25\lambda_3$$
$$\lambda_2 = 0.5\lambda_1 + 0.5\lambda_3 \qquad 0.5\lambda_1 = \lambda_2 - 0.5\lambda_3$$

Therefore $\lambda_2 = \lambda_3 = \lambda_1$ and $T(j) = 0.5\lambda_1 + 0.5\lambda_2 = \lambda_1$

Choosing Q1 as the reference queue with $\lambda_1 = \mu$, we get –

Relative Throughputs $\lambda_1 = \mu \quad \lambda_2 = \mu \quad \lambda_3 = \mu$

Visit Ratios $V_1 = 1 \quad V_2 = 1 \quad V_3 = 1$

Relative Utilizations $u_1 = 1 \quad u_2 = 0.5 \quad u_3 = 0.5$

Initialization $N_1 = 0 \quad N_2 = 0 \quad N_3 = 0$

Recursion

m=1

$$W_1(1) = 1 \quad W_2(1) = 0.5 \quad W_3(1) = 0.5 \qquad \lambda = \frac{1}{2} = 0.5$$
$$\lambda_1^*(1) = 0.5 \quad \lambda_2^*(1) = 0.5 \quad \lambda_3^*(1) = 0.5$$
$$N_1(1) = 0.5 \quad N_2(1) = 0.25 \quad N_3(1) = 0.25$$

m=2

$$W_1(2) = 1.5 \quad W_2(2) = 0.625 \quad W_3(2) = 0.625 \qquad \lambda = 0.7273$$
$$\lambda_1^*(2) = 0.7273 \quad \lambda_2^*(2) = 0.7273 \quad \lambda_3^*(2) = 0.7273$$
$$N_1(2) = 1.091 \quad N_2(2) = 0.4546 \quad N_3(2) = 0.4546$$

$$W_1(3)=2.091 \quad W_2(3)=0.7273 \quad W_3(3)=0.7273 \qquad \lambda=0.8461$$

m=3 $$\lambda_1^*(3)=0.8461 \quad \lambda_2^*(3)=0.8461 \quad \lambda_3^*(3)=0.8461$$

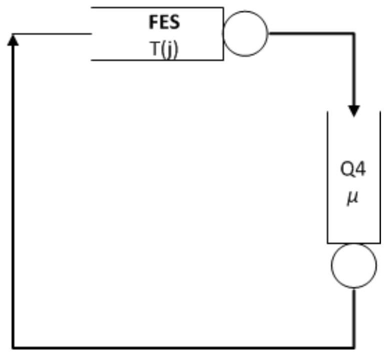
$$N_1(3)=1.7692 \quad N_2(3)=0.6154 \quad N_3(3)=0.6154$$

$$W_1(4)=2.7692 \quad W_2(4)=0.8077 \quad W_3(4)=0.8077 \quad \lambda=0.9123$$

m=4=*M* $$\lambda_1^*(4)=0.9123 \quad \lambda_2^*(4)=0.9123 \quad \lambda_3^*(4)=0.9123$$

$$N_1(4)=2.5263 \quad N_2(4)=0.7369 \quad N_3(4)=0.7369$$

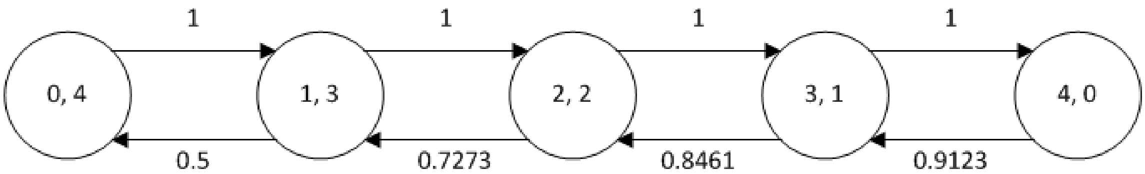
The State Dependent Service Rates for the corresponding FES would be



T(1)=0.5 *T*(2)=0.7273 **[5]**
T(3)=0.8461 *T*(4)=0.9123

(b) We can define the system state as (N_{FES}, N_{Q4}) where

$N_{FES} + N_{Q4} = 4$



The corresponding balance equations can be written to solve for the system state probabilities.

$$0.5p_{13} = p_{04} \qquad p_{13} = 2p_{04}$$
$$0.7273p_{22} = p_{13} \qquad p_{22} = 2.7499p_{04}$$
$$0.8461p_{31} = p_{22} \qquad p_{31} = 3.25p_{04}$$
$$0.9123p_{40} = p_{31} \qquad p_{04} = 3.5624p_{04}$$

Since $p_{04} + p_{13} + p_{22} + p_{31} + p_{40} = 1$, we get –

$$P_4 = p_{04} = 0.0796$$
$$P_3 = p_{13} = 0.1592$$
$$P_2 = p_{22} = 0.2189 \qquad \textbf{[5]}$$
$$P_1 = p_{31} = 0.2587$$
$$P_0 = p_{40} = 0.2836$$