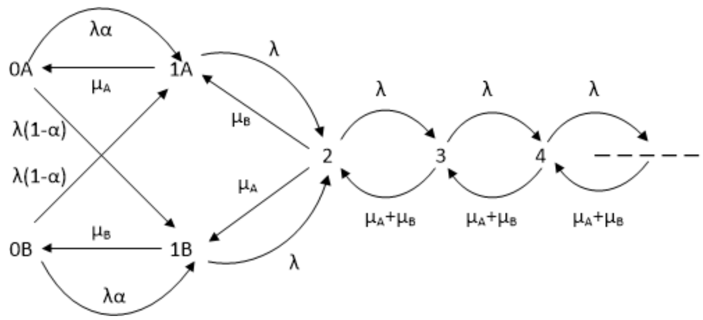
EE 633, Queueing Systems (2017-18F)
End Semester Exam, Solutions

1. (a) State Transition Diagram **[5]**



(b) Balance Equations

$$p_n\lambda = p_{n+1}(\mu_A+\mu_B) \quad n \ge 2$$
$$\lambda p_{0A} = \mu_A p_{1A} \qquad \lambda p_{0B} = \mu_B p_{1B}$$
$$(\lambda+\mu_A)p_{1A} = \lambda\alpha\, p_{0A} + \lambda(1-\alpha)\,p_{0B} + \mu_B p_2$$
$$(\lambda+\mu_B)p_{1B} = \lambda\alpha\, p_{0B} + \lambda(1-\alpha)\,p_{0A} + \mu_A p_2$$

[5]

(c) State Probability Calculations **[25]**

$$p_n = \left(\frac{\lambda}{\mu_A+\mu_B}\right)^{n-2} p_2 \quad n \ge 2 \qquad \Rightarrow \quad \sum_{n=2}^{\infty} p_n = \left(\frac{\mu_A+\mu_B}{\mu_A+\mu_B-\lambda}\right) p_2$$
$$\left(\lambda+(1-\alpha)\mu_A\right) p_{1A} - (1-\alpha)\mu_B p_{1B} = \mu_B p_2 \qquad (1)$$
$$-(1-\alpha)\mu_A p_{1A} + \left(\lambda+(1-\alpha)\mu_B\right) p_{1B} = \mu_A p_2 \qquad (2)$$

Adding (1) and (2), we get –

$$\lambda p_{1A} + \lambda p_{1B} = (\mu_A+\mu_B)p_2 \qquad (3)$$
$$p_{1A} + p_{1B} = \frac{\mu_A+\mu_B}{\lambda} p_2$$
$$\left(\lambda+\mu_A(1-\alpha)\right) p_{1A} - \mu_B(1-\alpha)\left[\frac{\mu_A+\mu_B}{\lambda} p_2 - p_{1A}\right] = \mu_B p_2$$
$$\left(\lambda+(1-\alpha)(\mu_A+\mu_B)\right) p_{1A} = \mu_B p_2\left[1+(1-\alpha)\left(\frac{\mu_A+\mu_B}{\lambda}\right)\right]$$

Note that (3) may be directly written

Therefore, $p_{1A} = \dfrac{\mu_B}{\lambda} p_2 \quad p_{1B} = \dfrac{\mu_A}{\lambda} p_2$ and $p_{0A} = p_{0B} = \dfrac{\mu_A\mu_B}{\lambda^2} p_2$

and, as obtained earlier $p_n = \left(\dfrac{\lambda}{\mu_A+\mu_B}\right)^{n-2} p_2 \quad n \ge 2$ with $\displaystyle\sum_{n=2}^{\infty} p_n = \left(\frac{\mu_A+\mu_B}{\mu_A+\mu_B-\lambda}\right) p_2$

Using the *Normalization Condition* $p_{0A} + p_{0B} + p_{1A} + p_{1B} + \sum_{n=2}^{\infty} p_n = 1$, we get –

$$p_2\left[\frac{2\mu_A\mu_B}{\lambda^2} + \frac{\mu_A+\mu_B}{\lambda} + \frac{\mu_A+\mu_B}{\mu_A+\mu_B-\lambda}\right] = 1$$

$$p_2\left[\frac{2\mu_A\mu_B}{\lambda^2} + \frac{(\mu_A+\mu_B)^2}{\lambda(\mu_A+\mu_B-\lambda)}\right] = 1 \quad \Rightarrow \quad p_2 = \frac{1}{\left[\frac{2\mu_A\mu_B}{\lambda^2} + \frac{(\mu_A+\mu_B)^2}{\lambda(\mu_A+\mu_B-\lambda)}\right]}$$

or $p_2 = \dfrac{\lambda^2(\mu_A+\mu_B-\lambda)}{2\mu_A\mu_B(\mu_A+\mu_B-\lambda)+(\mu_A+\mu_B)^2\lambda}$

Note the curious fact that α does not appear in the final answer for the state probabilities. [10 Bonus Marks for pointing this out.]

2.(a) L.T. of batch service time pdf $L_B(s) = 0.5L_\alpha(s)\left[1+L_\beta(s)\right]$

Mean of batch service time $\overline{X} = \alpha(1) + 0.5\beta(1)$

Second moment of batch service time $\overline{X^2} = \left[\alpha(2)+\alpha(1)\beta(1)+0.5\beta(2)\right]$

Offered Traffic $\rho = \lambda\overline{X} = \lambda\left[\alpha(1)+0.5\beta(1)\right]$

For a batch considered as one job, we get –

L.T.of the pdf of batch queueing delay $L_{Wqb}(s) = \dfrac{s(1-\rho)}{s-\lambda+\lambda L_B(s)}$

Mean batch queueing delay $W_{qb} = \dfrac{\lambda\overline{X^2}}{2(1-\rho)}$

Therefore,

Mean queueing delay $W_q = W_{qb} + \dfrac{1}{3}\alpha(1)$ **[10]**

L.T. of the pdf of the queueing delay $L_{Wq}(s) = \dfrac{1}{3}L_{Wqb}(s)\left[2+L_\alpha(s)\right]$ **[10]**

(b) Mean Queueing Delay for the second job in the batch = $W_{q2} = W_{qb} + \alpha(1)$ **[5]**

3. **(a)**

$$\sum_{n=1}^{M} nP(n_i = n) = 1 * P(n_i = 1) + 2 * P(n_i = 2) + 3 * P(n_i = 3) + ... + M * P(n_i = M)$$

$$= P(n_i = 1) + P(n_i = 2) + P(n_i = 3) + P(n_i = 4) + ... + P(n_i = M)$$

$$+ P(n_i = 2) + P(n_i = 3) + P(n_i = 4) + ... + P(n_i = M)$$

$$+ P(n_i = 3) + P(n_i = 4) + ... + P(n_i = M)$$ **[10]**

....

....

$$+ P(n_i = M)$$

Summing row by row, the result follows since,

$$P(n_i \geq 1) = P(n_i = 1) + P(n_i = 2) + P(n_i = 3) + P(n_i = 4) + ... + P(n_i = M)$$

$$P(n_i \geq 2) = P(n_i = 2) + P(n_i = 3) + P(n_i = 4) + ... + P(n_i = M)$$

....

....

(b) Solving this network directly, using either Convolution or MVA, we get the following. **Note that this has not been asked in the question and does not have to be done [10 Bonus Marks for doing this.]**

QUEUE	Q1	Q2	Q3	Q4
Visit Ratios:	1	2	2	2
Throughputs	0.2332	0.4664	0.4664	0.4664
Number in queue	0.2945	0.2945	0.7633	2.6478
Mean Time in queue	1.2626	0.6313	1.6364	5.6768

(A) To find the flow equivalent server (FES) where Q4 is the Target Queue, we short Q4 (i.e. make its service time 0) and then solve the network for the *actual throughputs* of Q1, Q2 and Q3. The flow through the short can then be calculated as ($0.5\lambda_2 + 0.5\lambda_3$).
Using MVA to solve the network with Q1, Q2 and Q3 (with Q4) shorted we get –

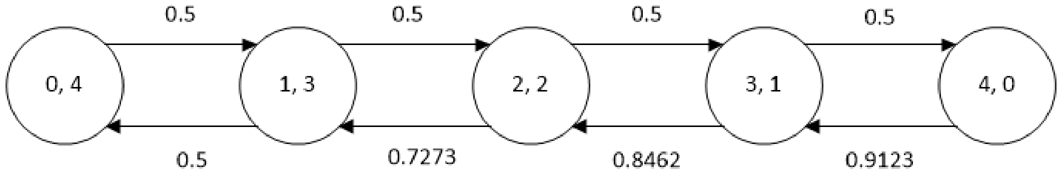
M	λ_1	λ_2	λ_3	$\lambda_{4,SC}$
1	0.25	0.5	0.5	0.5
2	0.3636	0.7273	0.7273	0.7273
3	0.4231	0.8462	0.8462	0.8462
4	0.4561	0.9123	0.9123	0.9123

The FES for Q1, Q2, and Q3 will then be given as –

Number in the FES	Service Rate
1	0.5
2	0.7273
3	0.8462
4	0.9123

[20]

(B) We use the service rates given above and the service rate of Q4 (i.e. 0.5) to draw the corresponding state transition diagram. The states are represented by (n_{FES}, n_{Q4})



Therefore,

$$0.5p_{1,3} = 0.5p_{0,4} \quad 0.7273p_{2,2} = 0.5p_{1,3} \quad 0.8462p_{3,1} = 0.5p_{2,2} \quad 0.9123p_{4,0} = 0.5p_{3,1}$$

$$p_{1,3} = p_{0,4} \quad p_{2,2} = 0.6875p_{0,4} \quad p_{3,1} = 0.4062p_{0,4} \quad p_{4,0} = 0.2226p_{0,4}$$

Using the Normalization Condition, we get, $p_{04} = 0.3015$

Therefore,

$$p_{0,4} = 0.3015 \quad p_{1,3} = 0.3015 \quad p_{2,2} = 0.2073 \quad p_{3,1} = 0.1225 \quad p_{4,0} = 0.0671$$ **[10]**

Therefore Mean Number in Q4 = 2.6476 (*Note this matches the MVA result*)

Mean Flow Rate through Q4 = $0.5\left(1 - p_{4,0}\right) = 0.4665$

(*This also matches the MVA result*)

Using Little's Formula, **Mean Transit Time through Q4** = 5.675 **[5]**

(*As obtained from MVA*)