EEE33 Queueing Systems (2011-12)
Mid Term Examination Solutions

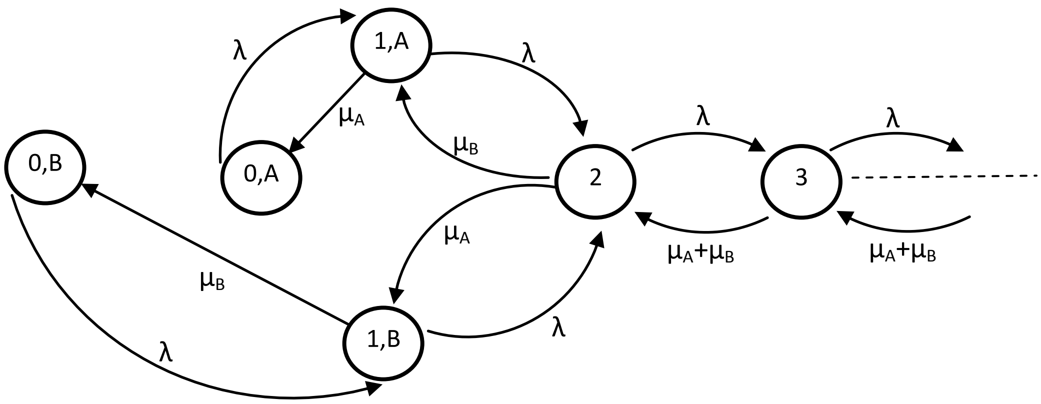
1. (a) States 2, 3,……. are defined normally. The other states are defined as follows.

{1,A} one customer in the system, Server A working

{1,B} one customer in the system, Server B working

{0,A} system empty, Server A idle for less time than Server B

{0,B} system empty, Server B idle for less time than Server A



(b) $$p_{1B} = \frac{\lambda}{\mu_B} p_{0B} \qquad p_{1A} = \frac{\lambda}{\mu_A} p_{0A} \qquad p_n = \left(\frac{\lambda}{\mu_A+\mu_B}\right)^{n-2} p_2 \qquad n = 2,3,4,\dots\dots$$

$$p_2 = \frac{\lambda}{\mu_A+\mu_B}\left(p_{1A}+p_{1B}\right) \qquad \sum_{n=2}^{\infty} p_n = \frac{\mu_A+\mu_B}{\mu_A+\mu_B-\lambda} p_2$$

$$(\lambda+\mu_A)p_{1A} = \mu_B p_2 + p_{0A}\lambda \qquad (\lambda+\mu_B)p_{1B} = \mu_A p_2 + p_{0B}\lambda$$

$$p_{1A} = \frac{\mu_B}{\lambda} p_2 \quad p_{0A} = \frac{\mu_A\mu_B}{\lambda^2} p_2 \quad p_{1B} = \frac{\mu_A}{\lambda} p_2 \quad p_{0B} = \frac{\mu_A\mu_B}{\lambda^2} p_2$$

$$p_n = \left(\frac{\lambda}{\mu_A+\mu_B}\right)^{n-2} p_2 \qquad n = 2,3,4,\dots.$$

Therefore, $$p_2 = \frac{1}{\left(2\frac{\mu_A\mu_B}{\lambda^2} + \frac{\mu_A+\mu_B}{\lambda} + \frac{\mu_A+\mu_B}{\mu_A+\mu_B-\lambda}\right)}$$

and

$$p_0 = \frac{2\mu_A\mu_B}{\lambda^2} p_2 \qquad p_1 = \frac{\mu_A+\mu_B}{\lambda} p_2 \qquad p_n = \left(\frac{\lambda}{\mu_A+\mu_B}\right)^{n-2} p_2 \qquad n = 2,3,4,\dots.$$

2. (a) $L_B(s) = 0.5L_{2B}(s) + 0.25L_{1B}(s) + 0.25L_{1B}(s)L_B(s)$

where $L_{1B}(s) = L_{2B}(s) = \frac{\mu}{s+\mu}$

Therefore,

$$L_B(s) = \frac{0.5L_{2B}(s) + 0.25L_{1B}(s)}{1 - 0.25L_{1B}(s)} = \frac{0.75\mu}{s + 0.75\mu}$$

(b) Note that the service time is exponentially distributed with parameter 0.75μ

Therefore,

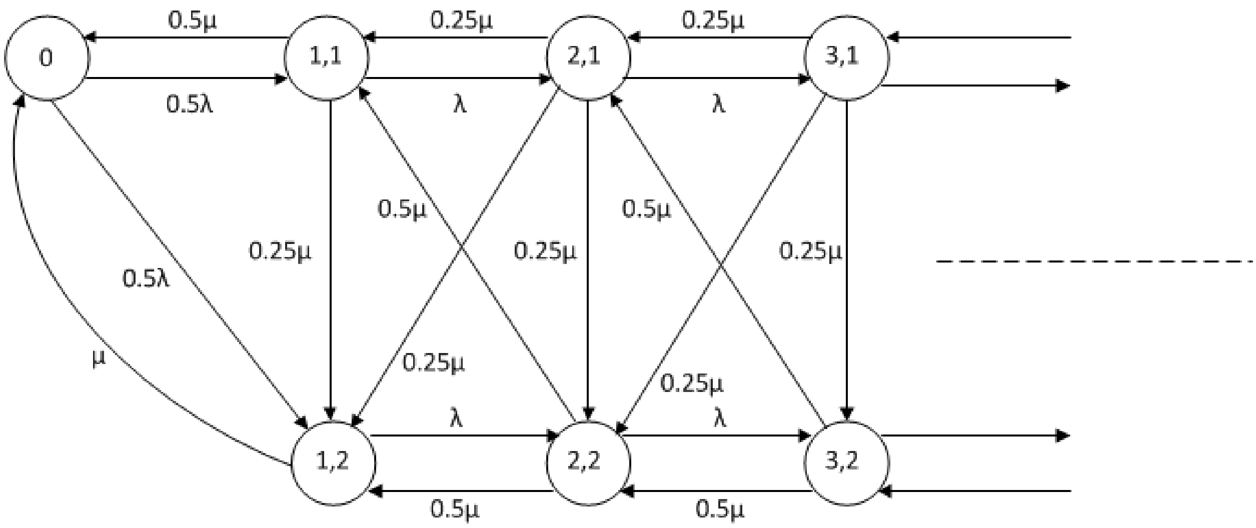
Mean = $\frac{1}{0.75\mu} = \frac{4}{3\mu}$

and

Second Moment = $\frac{2}{(0.75\mu)^2} = \frac{32}{9\mu^2}$

(Used the fact that an exponential distribution has mean $\overline{X}$ and second moment $\overline{X^2} = 2\left(\overline{X}\right)^2$. One can also find the second moment directly or from the Laplace Transform of the pdf.)

(c) State Transition Diagram for the usual definition of system states (*n*, *j*)



3. Considering an Idle-Busy cycle, the following three cases may arise –

(i) Two vacations followed by an idle, Probability = f_0^2

(ii) Two vacations, Probability = $f_0 f_j \quad j = 1, 2, \ldots\ldots\ldots$ Total Probability = $f_0(1 - f_0)$

(iii) One vacation, Probability = $f_j \quad j = 1, 2, \ldots\ldots\ldots$ Total Probability = $(1 - f_0)$

(a) Using $f_0 = L_V(\lambda)$

$$\overline{IP} = f_0^2\left(2\overline{V} + \frac{1}{\lambda}\right) + f_0(1 - f_0)(2\overline{V}) + (1 - f_0)\overline{V}$$

$$= f_0^2\frac{1}{\lambda} + f_0\overline{V} + \overline{V} = \frac{1}{\lambda}\left(f_0^2 + \lambda\overline{V}(1 + f_0)\right)$$

(b)

$$\overline{BP} = f_0^2\left(\frac{\overline{X}}{1 - \lambda\overline{X}}\right) + (1 + f_0)\sum_{j=1}^{\infty}\left(\frac{\overline{X}}{1 - \lambda\overline{X}}\right) j\, f_j$$

$$= \left(\frac{\overline{X}}{1 - \lambda\overline{X}}\right)\left[f_0^2 + \lambda\overline{V}(1 + f_0)\right]$$

P{server idle} = $\dfrac{\overline{IP}}{\overline{IP} + \overline{BP}} = 1 - \lambda\overline{X} = 1 - \rho$

(c) Mean Residual Time = $R = \lim_{t\to\infty}\left[\frac{1}{t}\sum_{i=1}^{M(t)}\frac{X_i^2}{2} + \frac{1}{t}\sum_{i=1}^{L(t)}\frac{V_i^2}{2}\right]$

$$R = \frac{1}{2}\lambda\overline{X^2} + \frac{(1 - \rho)(1 + f_0)}{\left(f_0^2 + \lambda\overline{V}(1 + f_0)\right)}\left(\frac{1}{2}\lambda\overline{V^2}\right)$$

(d) For the imbedded Markov Chain, we can write the following –

$$n_{i+1} = n_i + a_{i+1} - 1 \qquad n_i \ge 1 \qquad \text{Equilib. Prob.: } p_n$$

$$= a_{i+1} + j - 1 \qquad n_i = 0,\ j = 1, 2, \ldots\ldots\ldots \qquad \text{Equilib. Prob.: } p_0(1 + f_0) f_j$$

$$= a_{i+1} \qquad n_i = 0,\ j = 0 \qquad \text{Equilib. Prob.: } p_0 f_0^2$$

for j arrivals in the a vacation interval with probability f_j j=0,1,2,........

Taking expectations of both sides of the above, we get

$$\overline{n} = \overline{n} + \lambda\overline{X} - (1 - p_0) + p_0\left(\sum_{j=1}^{\infty}(j - 1) f_j (1 + f_0)\right)$$

$$= \overline{n} + \lambda\overline{X} - (1 - p_0) + p_0(1 + f_0)\left(\lambda\overline{V} - (1 - f_0)\right)$$

$$= \overline{n} + \lambda\overline{X} - 1 + p_0\left[(1 + f_0)\left(\lambda\overline{V} - 1 + f_0\right) + 1\right]$$

$$p_0 = \frac{1 - \lambda\overline{X}}{\left((1 + f_0)\lambda\overline{V} + f_0^2\right)}$$

and using $A(z) = L_B(\lambda - \lambda z) \quad F(z) = L_V(\lambda - \lambda z) \quad f_0 = L_V(\lambda)$

$$P(z) = A(z)\left[\frac{1}{z}\left(P(z) - p_0\right) + p_0(1+f_0)E\{z^{j-1}\} + p_0 f_0^2\right]$$
$$= A(z)\left[\frac{1}{z}\left(P(z) - p_0\right) + p_0 \frac{(1+f_0)}{z}\left(F(z) - f_0\right) + p_0 f_0^2\right]$$

Therefore,

$$P(z) = p_0 \frac{A(z)\left(1-(1+f_0)\left(F(z)-f_0\right) - f_0^2 z\right)}{A(z) - z} = p_0 \frac{A(z)\left(\left(1 - f_0^2 z\right) - (1+f_0)\left(F(z) - f_0\right)\right)}{A(z) - z}$$

Note that to do a sanity check, we can compare this result for infinitesimally small vacation lengths with a normal M/G/1 queue.