EEE33 Queueing Systems (2011-12)
Final Examination

Maximum Marks: 50 **Time: 3 hours**

1. **(a)** For the class 2 jobs, we can consider the system to be a simple M/G/1/2 system and solve for the state probabilities accordingly.

Considering the queue only for the Class 2 customers –

State probabilities at departure instants p_{di} i=0,1
State probabilities at arrival instants of those arrivals which actually enter the system p_{aci} i=0,1
State probabilities at arrival instants (all arrivals) p_{ai} i=0,1,2
Equilibrium State Probabilities p_i i=0,1,2 (Note $P_B = p_2$)

Transition probabilities at departure instants $p_{d,jk}$ j=0,1 and k=0,1

$$p_{d,00} = L_{B2}(\lambda_2) \quad p_{d,10} = L_{B2}(\lambda_2) \quad p_{d,01} = 1 - L_{B2}(\lambda_2) \quad p_{d,11} = 1 - L_{B2}(\lambda_2)$$

Balance Equation $p_{d0} = p_{d0}L_{B2}(\lambda_2) + p_{d1}L_{B2}(\lambda_2)$ and $p_{d0} + p_{d1} = 1$

Therefore, $p_{d0} = L_{B2}(\lambda_2) \quad p_{d1} = 1 - L_{B2}(\lambda_2)$
Using Kleinrock's Principle, $p_{ac0} = L_{B2}(\lambda_2) \quad p_{ac1} = 1 - L_{B2}(\lambda_2)$

Assuming P_{B2} as the blocking probability, we get –

$$p_{a0} = (1 - P_{B2})p_{ac0} = (1 - P_{B2})L_{B2}(\lambda_2) \quad p_{a1} = (1 - P_{B2})p_{ac1} = (1 - P_{B2})(1 - L_{B2}(\lambda_2)) \quad p_{a2} = P_{B2}$$

Using PASTA $p_i = p_{ai} \quad i = 0,1,2$

Traffic actually offered to the queue = $\rho_{c2} = \rho_2(1 - P_{B2})$ with $\rho_2 = \lambda_2 \bar{X}_2$

Therefore $p_0 = 1 - \rho_2(1 - P_{B2}) = (1 - P_{B2})L_{B2}(\lambda_2)$ or $P_{B2} = \dfrac{L_{B2}(\lambda_2) + \rho_2 - 1}{L_{B2}(\lambda_2) + \rho_2}$

$$\left.\begin{aligned} p_0 &= \frac{L_{B2}(\lambda_2)}{L_{B2}(\lambda_2) + \rho_2} \\ p_1 &= \frac{1 - L_{B2}(\lambda_2)}{L_{B2}(\lambda_2) + \rho_2} \\ p_2 = P_{B2} &= \frac{L_{B2}(\lambda_2) + \rho_2 - 1}{L_{B2}(\lambda_2) + \rho_2} \end{aligned}\right\} \text{State Probabilities for Class 2 jobs}$$

(b) Let α = P{no Class 2 arrivals in a Class 2 service time}

$$= \int_0^\infty e^{-\lambda_2 x} b_2(x)dx = L_{B2}(\lambda_2)$$

$$\overline{BP2} = \sum_{j=1}^{\infty} j\overline{X_2}\alpha(1-\alpha)^{j-1} = \frac{\overline{X_2}}{\alpha} = \frac{\overline{X_2}}{L_{B2}(\lambda_2)}$$

and

$L_{BP2}(s)$ = E{ $e^{-s(BP2)}$ }= $\sum_{n=1}^{\infty} L_{B2}^{n}(s)(1-\alpha)^{n-1}\alpha = \left(\frac{\alpha}{1-\alpha}\right)\frac{(1-\alpha)L_{B2}(s)}{1-(1-\alpha)L_{B2}(s)}$

Therefore $$L_{BP2}(s) = \frac{L_{B2}(\lambda_2)L_{B2}(s)}{1-\left[1-L_{B2}(\lambda_2)\right]L_{B2}(s)} = \frac{L_{B2}(\lambda_2)L_{B2}(s)}{\left[1-L_{B2}(s)\right]+L_{B2}(\lambda_2)L_{B2}(s)}$$

Note that we can also get $\overline{BP2}$ by differentiating the above and evaluating it at s=0 (add a minus sign!)

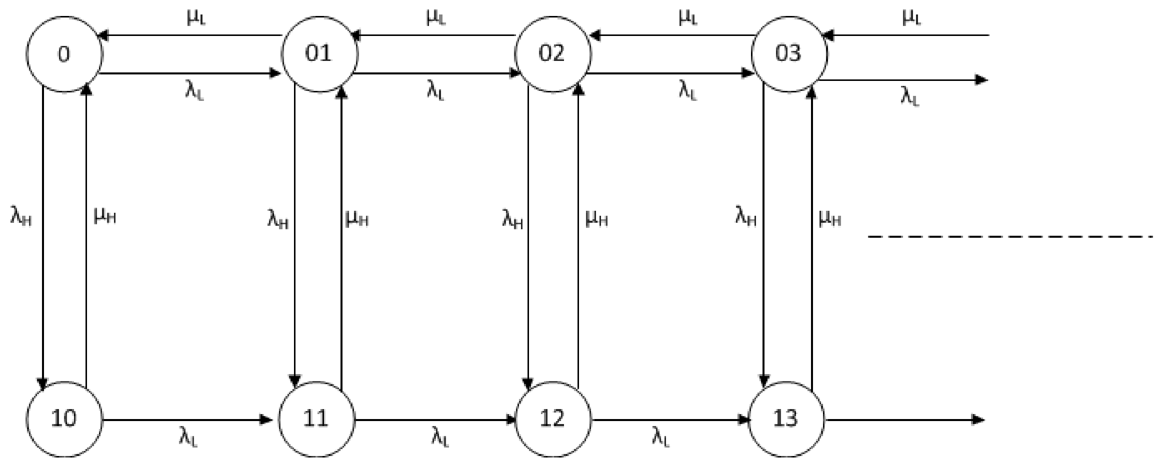
(c) $\overline{T} = \overline{X_1} + \lambda_2\overline{X_1}\left(\overline{BP2}\right) = \overline{X_1}\left(1+\frac{\rho_2}{L_{B2}(\lambda_2)}\right)$

and

$$\begin{aligned} L_T(s) = E\{e^{-sT}\} &= \sum_{n=0}^{\infty} E\left\{e^{-sX_1}L_{BP2}^{n}(s)\frac{(\lambda_1 X_1)^n}{n!}e^{-\lambda_1 X_1}\right\} \\ &= E\left\{\sum_{n=0}^{\infty} e^{-(s+\lambda_1)X_1}\frac{(\lambda_1 X_1 L_{BP2}(s))^n}{n!}\right\} \\ &= E\left\{e^{-(s+\lambda_1-\lambda_1 L_{BP2}(s))X_1}\right\} \\ &= L_{B1}(s+\lambda_1-\lambda_1 L_{BP2}(s)) \end{aligned}$$

2. Representing the system state as (n_H,n_L), we can draw the state transition diagram of the system as shown below.

(a)



$p_{10}(\lambda_L + \mu_H) = p_0\lambda_H$ $\qquad$ $p_{10} = \frac{\lambda_H}{\lambda_L + \mu_H} p_0$

$$(\lambda_L + \lambda_H) p_0 = p_{01}\mu_L + p_{10}\mu_H \qquad p_{01} = \rho_L \left(1 + \frac{\lambda_H}{\lambda_L + \mu_H}\right) p_0$$

Define $P_{0x} = \sum_{i=1}^{\infty} p_{0i}$ and $P_{1x} = \sum_{i=0}^{\infty} p_{1i}$ $\Rightarrow$ $p_0 + P_{0x} + P_{1x} = 1$

$$\left.\begin{aligned} p_{01} &= \rho_L (p_0 + p_{10}) \\ p_{02} &= \rho_L (p_{01} + p_{11}) \\ p_{03} &= \rho_L (p_{02} + p_{12}) \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad \begin{aligned} P_{0x} &= \rho_L (P_{0x} + P_{1x}) + \rho_L p_0 \\ P_{0x} &= \frac{\rho_L}{1-\rho_L}(P_{1x} + p_0) = \frac{\rho_L}{1-\rho_L}(1 - P_{0x}) \end{aligned} \quad \text{or} \quad P_{0x} = \rho_L$$

$$\lambda_H (p_0 + p_{01} + p_{02} + \ldots\ldots) = \mu_H (p_{10} + p_{11} + \ldots\ldots\ldots)$$

$$P_{1x} = \rho_H (p_0 + P_{0x}) = \rho_H (1 - P_{1x}) \qquad \text{or} \qquad P_{1x} = \frac{\rho_H}{1+\rho_H}$$

and $$p_0 = \frac{1}{(1+\rho_H)} - \rho_L$$

(b) P{server is idle} = $p_0 = \frac{1}{(1+\rho_H)} - \rho_L$

(c) P{Blocking for high priority jobs} = $P_{1x} = \frac{\rho_H}{1+\rho_H}$

(d) P{finding one job in system} = $p_{01} + p_{10}$

$$= \left(\rho_L + (1+\rho_L)\frac{\lambda_H}{\lambda_L + \mu_H}\right) p_0 = \left(\frac{\lambda_H + \rho_L(\lambda_H + \lambda_L + \mu_H)}{(\lambda_L + \mu_H)}\right) p_0$$

$$= \left(\frac{\lambda_H + \rho_L(\lambda_H + \lambda_L + \mu_H)}{(\lambda_L + \mu_H)}\right)\left(\frac{1}{(1+\rho_H)} - \rho_L\right)$$

3. **(a)** L.T. of batch service time pdf $L_B(s) = 0.5 L_\alpha(s)\left[1 + L_\beta(s)\right]$

Mean of batch service time $\overline{X} = \alpha(1) + 0.5\beta(1)$

Second moment of batch service time $\overline{X^2} = \left(\alpha(2) + \alpha(1)\beta(1) + 0.5\beta(2)\right)$

Offered Traffic $\rho = \lambda \overline{X} = \lambda\left[\alpha(1) + 0.5\beta(1)\right]$

For a batch considered as one job, we get –

L.T.of the pdf of batch queueing delay $L_{Wqb}(s) = \frac{s(1-\rho)}{s - \lambda + \lambda L_B(s)}$

Mean batch queueing delay $W_{qb} = \frac{\lambda \overline{X^2}}{2(1-\rho)}$

Therefore,

Mean queueing delay $W_q = W_{qb} + \frac{1}{3}\alpha(1)$

L.T. of the pdf of the queueing delay $L_{Wq}(s) = \frac{1}{3} L_{Wqb}(s)\left[2 + L_\alpha(s)\right]$

(b) Mean Queueing Delay for the second job in the batch = $W_{q2} = W_{qb} + \alpha(1)$

4. The flow balance equations are as follows.

$$\lambda_1 = \lambda + \lambda_3 + \lambda_4$$
$$\lambda_2 = 0.5[0.5\lambda_1 + 0.1\lambda_2] = 0.25\lambda_1 + 0.05\lambda_2 \quad \Rightarrow \quad \lambda_1 = 3.8\lambda_2$$
$$\lambda_3 = 2\lambda + \lambda_2 \qquad \lambda_4 = 0.4\lambda_2$$

Therefore, $3.8\lambda_2 = \lambda + 2\lambda + \lambda_2 + 0.4\lambda_2$ $\qquad \lambda_2 = 1.25\lambda$

$\tilde{\lambda} = (4.75\lambda, 1.25\lambda, 3.25\lambda, 0.5\lambda)$ and $\tilde{\rho} = (4.75\rho, 2.5\rho, 6.5\rho, 0.5\rho)$ for $\rho=\lambda/\mu$

(a) For system to be stable, we need $6.5\rho<1$ or $\lambda<0.1538\mu$

(b) For λ=0.1 and μ=1, we have

$\tilde{\lambda} = (0.475, 0.125, 0.325, 0.05)$ $\qquad \tilde{\rho} = (0.475, 0.25, 0.65, 0.05)$

$\tilde{N} = (0.905, 0.333, 1.857, 0.053)$

Total Number in system = 3.148

Therefore, Mean Time spent in System = 3.148/0.3 = 10.49

The individual delays through Q1-Q4 are given below (useful later in (c))

$\tilde{W} = (1.9053, 2.66666, 5.71385, 1.06)$

(c) For λ=0.1 and μ=1, we set the input from **A** to zero (i.e. only considering the inputs from **B)** and calculate the corresponding flows for each queue using flow balance

$$\lambda_1 = \lambda_3 + \lambda_4$$
$$\lambda_2 = 0.5[0.5\lambda_1 + 0.1\lambda_2] = 0.25\lambda_1 + 0.05\lambda_2 \quad \Rightarrow \quad \lambda_1 = 3.8\lambda_2$$
$$\lambda_3 = 0.2 + \lambda_2 \qquad \lambda_4 = 0.4\lambda_2$$

Therefore, $3.8\lambda_2 = 0.2 + \lambda_2 + 0.4\lambda_2$ $\qquad \lambda_2 = 0.2/2.4 = 0.08333$

$\tilde{\lambda} = (0.31667, 0.08333, 0.28333, 0.033333)$

Visit Ratios: $\tilde{V} = (1.58333, 0.41667, 1.41667, 0.16667)$

Mean Transit Time for Job entering at B = $\sum_{i=1}^{4} V_i W_i = 12.39909$

5. **(a)** Since Q4 is the designated sub-network, for computation of the FES, we need to redraw the network with Q4 shorted. We then compute $T(j)=0.5\lambda_2^*(j)$ as the throughput through that short for $j=1,2,\ldots,M$ where j is the number of jobs circulating in the network. (Note $M=4$)

$$\lambda_1 = \lambda_3 + 0.5\lambda_2 \qquad \lambda_2 = \lambda_3 = 0.5[\lambda_1 + 0.5\lambda_2]$$

Therefore $\lambda_2 = \lambda_3 = \frac{2}{3}\lambda_1$

Choosing Q1 as the reference queue with $\lambda_1 = \mu$, we get –

Relative Throughputs $\qquad \lambda_1 = \mu \quad \lambda_2 = \frac{2}{3}\mu \quad \lambda_3 = \frac{2}{3}\mu$

Visit Ratios $V_1 = 1 \quad V_2 = \frac{2}{3} \quad V_3 = \frac{2}{3}$

Relative Utilizations $u_1 = 1 \quad u_2 = \frac{4}{3} \quad u_3 = \frac{2}{3}$

Initialization $N_1 = 0 \quad N_2 = 0 \quad N_3 = 0$

Recursion

j=1

$W_1(1) = 1 \quad W_2(1) = 2 \quad W_3(1) = 1 \qquad \lambda = \frac{1}{3} = 0.33333$

$\lambda_1^*(1) = 0.33333 \quad \lambda_2^*(1) = 0.22222 \quad \lambda_3^*(1) = 0.22222$

$N_1(1) = 0.33333 \quad N_2(1) = 0.44444 \quad N_3(1) = 0.22222$

j=2

$W_1(2) = 1.33333 \quad W_2(2) = 2.88889 \quad W_3(2) = 1.22222 \qquad \lambda = 0.49091$

$\lambda_1^*(2) = 0.49091 \quad \lambda_2^*(2) = 0.32727 \quad \lambda_3^*(2) = 0.32727$

$N_1(2) = 0.65455 \quad N_2(2) = 0.94545 \quad N_3(2) = 0.4$

j=3

$W_1(3) = 1.65455 \quad W_2(3) = 3.89091 \quad W_3(3) = 1.4 \qquad \lambda = 0.57895$

$\lambda_1^*(3) = 0.57895 \quad \lambda_2^*(3) = 0.38597 \quad \lambda_3^*(3) = 0.38597$

$N_1(3) = 0.9579 \quad N_2(3) = 1.50175 \quad N_3(3) = 0.54035$

j=4=M

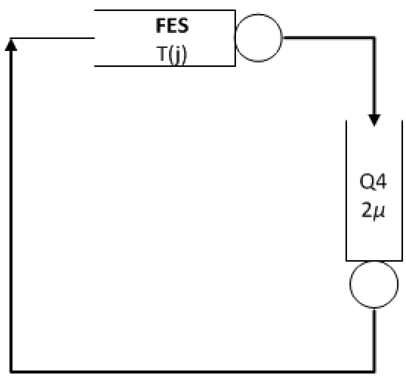
$W_1(4) = 1.9579 \quad W_2(4) = 5.00351 \quad W_3(4) = 1.54035 \qquad \lambda = 0.63287$

$\lambda_1^*(4) = 0.63287 \quad \lambda_2^*(4) = 0.42191 \quad \lambda_3^*(4) = 0.42191$

$N_1(4) = 1.23908 \quad N_2(4) = 2.11103 \quad N_3(4) = 0.6499$

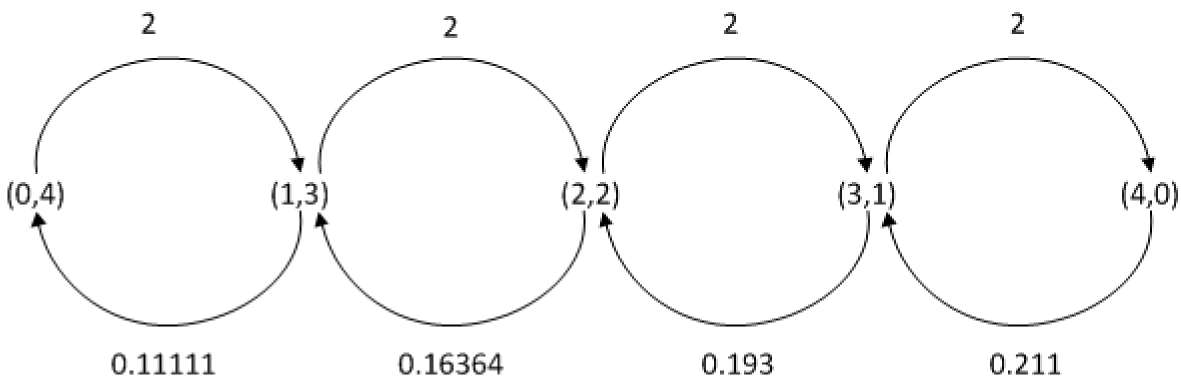
The State Dependent Service Rates for the corresponding FES would be

T(1)=0.11111 *T*(2)=0.16364
T(3)=0.193 *T*(4)=0.211



(b) We can define the system state as (N_{FES}, N_{Q4}) where

$N_{FES} + N_{Q4} = 4$



The corresponding balance equations can be written to solve for the system state probabilities.

$p_{13} = 18.0002\, p_{04}$ $\quad$ $p_{22} = 12.222\, p_{13} = 220.02\, p_{04}$

$p_{31} = 10.3627\, p_{22} = 2280\, p_{04}$ $\quad$ $p_{40} = 9.4787\, p_{31} = 21611.27\, p_{04}$

Since $p_{04} + p_{13} + p_{22} + p_{31} + p_{40} = 1$, we get –

$$P_4 = p_{04} = 0.41442 \times 10^{-4} = 0.000041$$
$$P_3 = p_{13} = 0.74604 \times 10^{-3} = 0.000746$$
$$P_2 = p_{22} = 0.009118$$
$$P_1 = p_{31} = 0.094488$$
$$P_0 = p_{40} = 0.895618$$