EC633 Queueing Systems (2009-10-I)
Mid Term Examination
Solutions

1. $p_0(t+\Delta t) = (1-\lambda\Delta t)\,p_0(t) + \mu\Delta t p_1(t)$ $\qquad \dfrac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t)$

$p_1(t+\Delta t) = (1-\mu\Delta t)\,p_1(t) + p_0(t)\lambda\Delta t$ $\qquad \dfrac{dp_1(t)}{dt} = -\mu p_1(t) + \lambda p_0(t)$

with $p_0(0^+)=1$, $p_1(0^+)=0$ and $p_0(t) + p_1(t) = 1$

Therefore, $$\frac{dp_0(t)}{dt} + (\lambda+\mu)\,p_0(t) = \mu$$

$$p_0(t) = \frac{\int e^{(\lambda+\mu)t}\mu dt + C}{e^{(\lambda+\mu)t}} = \frac{\mu}{\lambda+\mu} + Ce^{-(\lambda+\mu)t}$$

Using $p_0(0^+)=1$, we get $1 = \dfrac{\mu}{\lambda+\mu} + C$ or $C = \dfrac{\lambda}{\lambda+\mu}$

So $$p_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

and $$p_1(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

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2 **(a)** A State Transition Diagram for this system is given below. Note that states 2, 3,…∞ are normally defined. The other states are defined as follows –

{1,A} one customer in the system, Server A working
{1,B} one customer in the system, Server B working
{0,A} system empty, Server A idle for less time than Server B
{0,B} system empty, Server B idle for less time than Server A

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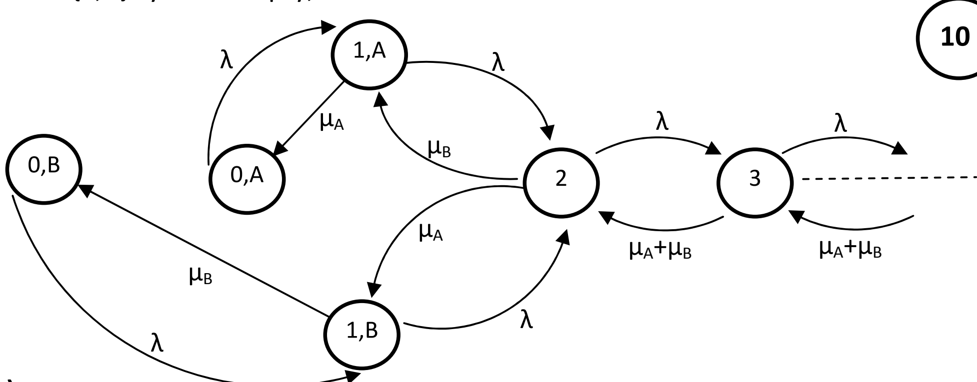


(b)

$$p_{1B} = \frac{\lambda}{\mu_B}p_{0B} \qquad p_{1A} = \frac{\lambda}{\mu_A}p_{0A} \qquad p_n = \left(\frac{\lambda}{\mu_A+\mu_B}\right)^{n-2} p_2 \qquad n = 2,3,4,.......$$

$$p_2 = \frac{\lambda}{\mu_A+\mu_B}(p_{1A}+p_{1B}) \qquad \sum_{n=2}^{\infty} p_n = \frac{\mu_A+\mu_B}{\mu_A+\mu_B-\lambda}p_2$$

$$(\lambda+\mu_A)p_{1A} = \mu_B p_2 + p_{0A}\lambda \qquad (\lambda+\mu_B)p_{1B} = \mu_A p_2 + p_{0B}\lambda$$

$$p_{1A}=\frac{\mu_B}{\lambda}p_2 \quad p_{0A}=\frac{\mu_A\mu_B}{\lambda^2}p_2 \quad p_{1B}=\frac{\mu_A}{\lambda}p_2 \quad p_{0B}=\frac{\mu_A\mu_B}{\lambda^2}p_2$$

$$p_n=\left(\frac{\lambda}{\mu_A+\mu_B}\right)^{n-2}p_2 \qquad n=2,3,4,....$$

with $p_2=\dfrac{1}{\left(2\dfrac{\mu_A\mu_B}{\lambda^2}+\dfrac{\mu_A+\mu_B}{\lambda}+\dfrac{\mu_A+\mu_B}{\mu_A+\mu_B-\lambda}\right)}$ **10**

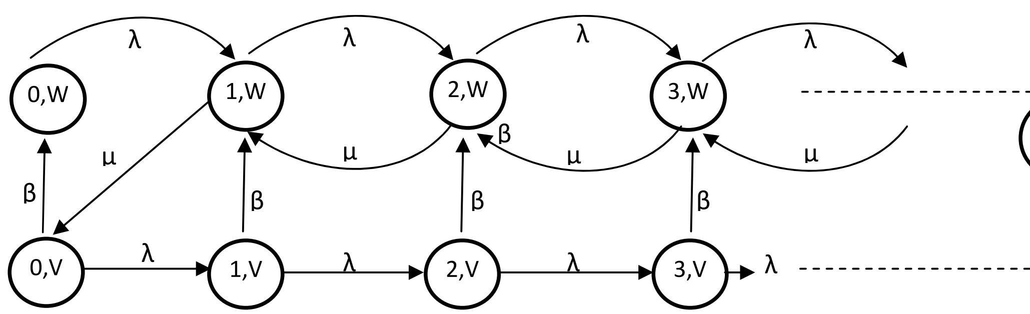
$$p_0=\frac{2\mu_A\mu_B}{\lambda^2}p_2 \qquad p_1=\frac{\mu_A+\mu_B}{\lambda}p_2 \qquad p_n=\left(\frac{\lambda}{\mu_A+\mu_B}\right)^{n-2}p_2 \qquad n=2,3,4,....$$

(c) P{server A not working} = $p_{0A}+p_{0B}+p_{1B}=\left(\frac{2\mu_A\mu_B}{\lambda^2}+\frac{\mu_A}{\lambda}\right)p_2$

P{server B not working} = $p_{0A}+p_{0B}+p_{1A}=\left(\frac{2\mu_A\mu_B}{\lambda^2}+\frac{\mu_B}{\lambda}\right)p_2$

$$\frac{\text{Penalty to Server A}}{\text{Penalty to Server B}}=\frac{\lambda\mu_A+2\mu_B\mu_A}{\lambda\mu_B+2\mu_A\mu_B}$$ **5**

3. **(a)** The State Transition Diagram is given below.



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The states (1,W), (2,W) ……etc. are the normal states with the server working. The states (*k*,V) are ones where the server is on its single vacation and *k* arrivals are in the system. The state (0,W) is the state where the server has completed its vacation (with no arrivals) and is now waiting for new arrivals to come in order to start service.

(b) We can write the following balance equations for this

$$\beta p_{0V}=\lambda p_{0W} \qquad (\lambda+\beta)p_{0V}=\mu p_{1W} \qquad (\lambda+\mu)p_{1W}=\lambda p_{0W}+\mu p_{2W}+\beta p_{1V}$$
$$(\lambda+\beta)p_{1V}=\lambda p_{0V} \qquad (\lambda+\beta)p_{2V}=\lambda p_{1V} \text{ .... } (\lambda+\beta)p_{(n+1)V}=\lambda p_{nV} \text{ ....}$$
$$\lambda(p_{1W}+p_{1V})=\mu p_{2W} \text{ .... } \lambda(p_{nW}+p_{nV})=\mu p_{(n+1)W} \text{ ....}$$

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(c) These give

$$p_{0W} = \frac{\beta}{\lambda} p_{0V} \quad p_{1W} = \frac{\lambda+\beta}{\mu} p_{0V} \quad p_{1V} = \left(\frac{\lambda}{\lambda+\beta}\right) p_{0V} \; \ldots\ldots\ldots \; p_{nV} = \left(\frac{\lambda}{\lambda+\beta}\right)^n p_{0V} \; \ldots\ldots$$

Therefore, $P(V) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda+\beta}\right)^n p_{0V} = \frac{\lambda+\beta}{\beta} p_{0V}$ where *P*(*V*) is the probability that the server is on vacation. To find *P*(*V*) we still need to find p_{0V} which may be done using the normalization condition.

Note that $\lambda(p_{nW} + p_{nV}) = \mu p_{(n+1)W}$ for *n*=1,2,............. Summing these from *n*=1 to ∞ and denoting $P(W) = \sum_{n=1}^{\infty} p_{nW}$ for notational convenience, we get

$$\lambda P(W) + \lambda\left[\frac{\lambda+\beta}{\beta} - 1\right] p_{0V} = \mu P(W) - \mu \frac{\lambda+\beta}{\mu} p_{0V}$$

$$(\mu-\lambda)P(W) = \left(\frac{\lambda^2}{\beta} + \lambda + \beta\right) p_{0V} \qquad \Rightarrow P(W) = \frac{\lambda^2 + \lambda\beta + \beta^2}{(\mu-\lambda)\beta} p_{0V}$$

Normalization implies that $P(W) + P(V) + p_{0W} = 1$

Therefore,

$$p_{0V}\left(\frac{\lambda^2 + \lambda\beta + \beta^2}{(\mu-\lambda)\beta} + \frac{\lambda+\beta}{\beta} + \frac{\beta}{\lambda}\right) = 1$$

$$p_{0V} = \frac{\lambda\beta(\mu-\lambda)}{\mu\left(\lambda^2 + \lambda\beta + \beta^2\right)}$$

Using these the required probability *P*(*V*) may be found as

$$\boxed{P(V) = \frac{\lambda(\lambda+\beta)(\mu-\lambda)}{\mu(\lambda^2 + \lambda\beta + \beta^2)}}$$

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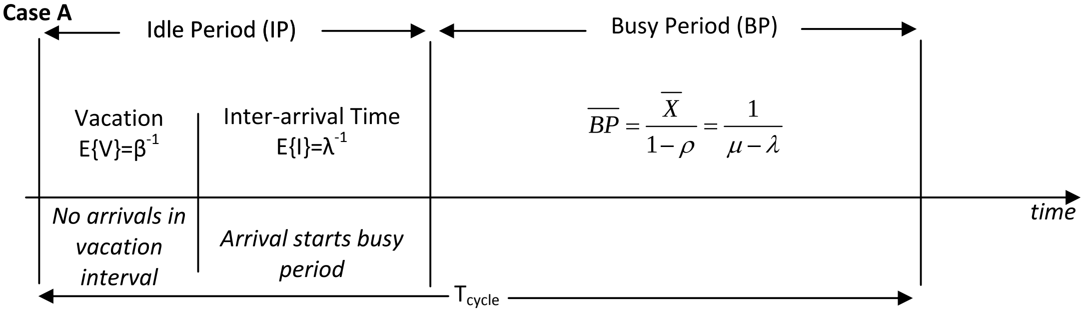
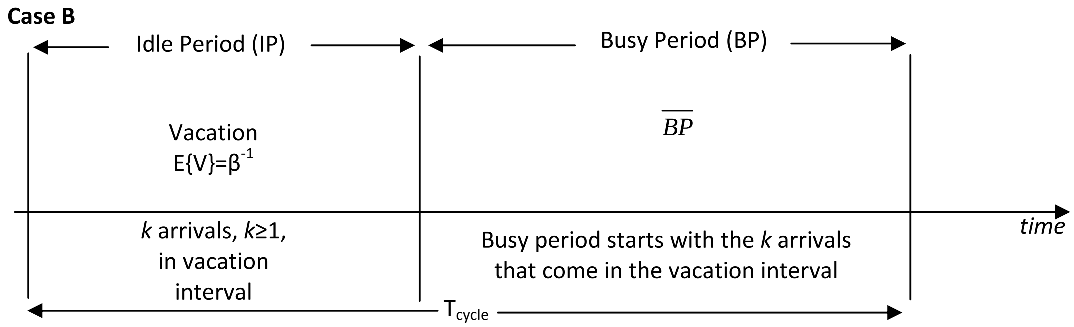
Alternative Approach for finding *P*(*V*) directly

We consider the Busy/Idle cycles for the server that will be observed. There are two different possibilities for this.

Case A





Case A

Probability = $\int_0^\infty e^{-\lambda v}\beta e^{-\beta v}dv = \frac{\beta}{\lambda+\beta}$

$$E\{T_{cycle}\}\,|Case\ A\} = \frac{1}{\beta}+\frac{1}{\lambda}+\frac{1}{\mu-\lambda}$$

Case B

Probability = $1-\frac{\beta}{\lambda+\beta} = \frac{\lambda}{\lambda+\beta}$

$$E\{T_{cycle}\}\,|Case\ B\} = \frac{1}{\beta}+\frac{\lambda/\beta}{\left(1-\frac{\beta}{\lambda+\beta}\right)}\left[\frac{1}{\mu-\lambda}\right] = \frac{1}{\beta}+\frac{(\lambda+\beta)}{\beta}\left[\frac{1}{\mu-\lambda}\right]$$

Note that a vacation interval where there are one or more arrivals will have a mean length of $\frac{1/\beta}{\left(1-\frac{\beta}{\lambda+\beta}\right)}$ and the average number of arrivals in that interval will be $\frac{\lambda/\beta}{\left(1-\frac{\beta}{\lambda+\beta}\right)}$. Also note that each one of those arrivals will cause a busy period of $\frac{\overline{X}}{1-\rho} = \frac{1}{\mu-\lambda}$ and that each of these busy periods will be added up to give the overall busy period.

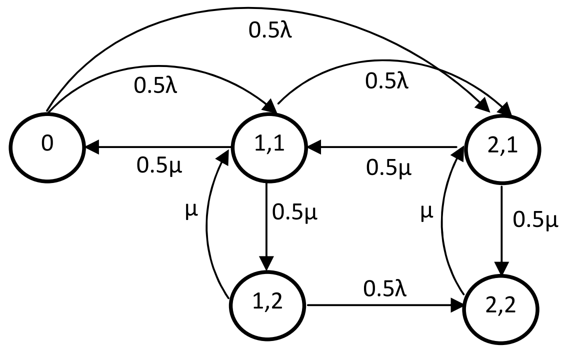
So $$E\{T_{cycle}\} = \left(\frac{1}{\beta}+\frac{\beta}{\lambda(\lambda+\beta)}\right)+\left(\frac{1}{\mu-\lambda}\right)\left(\frac{\beta}{(\lambda+\beta)}+\frac{\lambda}{\beta}\right) = \frac{\left(\lambda^2+\lambda\beta+\beta^2\right)}{\lambda\beta(\lambda+\beta)}\left[1+\frac{\lambda}{\mu-\lambda}\right]$$

$$= \frac{\left(\lambda^2+\lambda\beta+\beta^2\right)}{\lambda\beta(\lambda+\beta)}\left[\frac{\mu}{\mu-\lambda}\right]$$

$$P(V) = \frac{1/\beta}{E\{T_{cycle}\}} == \frac{\lambda(\lambda+\beta)(\mu-\lambda)}{\mu(\lambda^2+\lambda\beta+\beta^2)}$$

4. **(a)** $$L_B(s) = \frac{0.5\mu}{s+\mu}\sum_{n=0}^{\infty}\left[0.5\frac{\mu^2}{(s+\mu)^2}\right]^n = \frac{0.5\mu(s+\mu)}{s^2+2s\mu+0.5\mu^2}$$ **5**

(b) System State (n, k) n jobs in system, current job served in stage k



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(c) The balance equations are –

$$0.5\mu p_{11} = \lambda p_0 \qquad (0.5\lambda+\mu)p_{12} = 0.5\mu p_{11}$$
$$\mu p_{21} = \mu p_{22} + 0.5\lambda p_{11} + 0.5\lambda p_0$$
$$\mu p_{22} = 0.5\mu p_{21} + 0.5\lambda p_{12}$$
$$\rho = \frac{\lambda}{\mu}$$

Solving -

$$p_{11} = 2\rho p_0 \qquad p_{12} = \frac{2\rho}{\rho+2}p_0$$
$$p_{21} = \frac{\rho}{\rho+2}\left(2\rho^2+7\rho+2\right)p_0$$
$$p_{22} = \frac{\rho}{\rho+2}\left(\rho^2+4.5\rho+1\right)p_0$$

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Using the normalization condition

$$p_0\left(1+2\rho+\frac{2\rho}{\rho+2}+\frac{\rho}{\rho+2}\left(2\rho^2+7\rho+2\right)+\frac{\rho}{\rho+2}\left(\rho^2+4.5\rho+1\right)\right)=1$$
$$p_0 = \frac{\rho+2}{3\rho^3+13.5\rho^2+10\rho+2}$$

Using p_0 from above, the individual state probabilities can be found.

(d) P{arriving batch denied entry}

$$= p_{21}+p_{22}+0.5(p_{11}+p_{12}) = \left(\frac{\rho}{\rho+2}\right)\left(3\rho^2+13.5\rho+9\right)p_0$$ **5**